Optimal sensor location for inverse heat conduction problem in multilayered building walls

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Abstract - This article deals with the optimal sensor positioning in a two-layer slab to retrieve the thermal conductivity of each layer. Three algorithms are evaluated to maximize the D–optimum criterion quantifying the identification accuracy. Results show that the exchange algorithm is an efficient approach to determine a local optimum design with a minimum computational cost. In addition, the strategy based on convex relaxation to place the sensor provides complementary information to the experimenter.

Nomenclature

c	volumetric heat capacity, $\mathrm{J\,.\,m^{-3}\,.\,K^{-1}}$
$\mathcal{D}$	experimental design
h	surface heat transf. coeff., $\mathrm{W\,.\,m^{-2}\,.\,K^{-1}}$
$J\,,\,\xi$	set of label position
k	thermal conductivity, $\mathrm{W\,.\,m^{-1}\,.\,K^{-1}}$
ℓ	wall length, m
$\mathbf{M}$	information matrix
q	radiation flux, $\mathrm{W\,.\,m^{-2}}$
x	spatial coordinate, m
t	time, s
T	wall temperature, K
w	sensor indicator variable, $\big[-\big]$
Greek symbols	
θ	sensitivity coefficients, $\big[-\big]$
Φ	D-optimum criterion, $\big[-\big]$
Ω_t	time domain, s
Ω_χ	candidate location domain, m
Ω_x	space domain, m
Index and exponent	
L	left boundary
R	right boundary
∞	ambient air

1. Introduction

In France, the average increase of the building stock scales with $1\,\%$, highlighting a crucial environmental issue on building retrofitting. To efficiently plan such actions, *in-situ* diagnosis are required to determine the uncertain thermophysical properties of the layers composing the walls. Such inverse problem can be solved using experimental observations of temperature inside the wall [1]. To maximize the accuracy of the estimates, it is crucial to determine the optimal experiment design (OED) before carrying the experiments. In this article, the OED is explored with respect to the sensor positioning, considering a thermal conductivity inverse heat conduction problem in a two-layer slab submitted to climatic boundary conditions. Three algorithms are investigated and discussed to determine the OED, *i.e.* the optimal positions of the sensors inside the wall to determine the diffusivity of each layer.

2. Mathematical model

The investigations focus on the heat transfer through a multi-layer building wall as illustrated in Figure 1(a). The space domain is $\Omega_x = \big[\,0\,,\,\ell\,\big]$, where $\ell\,\big[\,\mathrm{m}\,\big]$ is the length of the wall. We

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denote by ℓ_1 and ℓ_2 the length of each layer. By convention, $\ell = \ell_1 + \ell_2$. The phenomena occur over the interval $\Omega_t = \left[0, t_f\right]$, where $t_f \left[\mathrm{s}\right]$ is the time horizon. The one-dimensional heat transfer governing equation is:

$$c_i \frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left(k_i \frac{\partial T}{\partial x}\right), \quad \forall x \in \Omega_x, \quad t \in \Omega_t, \quad \forall i \in \left\{1, 2\right\}, \tag{1}$$

where $T \left[\mathrm{K}\right]$ is the temperature inside the wall and the material properties of each layer are the thermal conductivity $k_i \left[\mathrm{W \,.\, m^{-1} \,.\, K^{-1}}\right]$ and the volumetric heat capacity $c_i \left[\mathrm{J \,.\, m^{-3} \,.\, K^{-1}}\right]$. At the interface between the wall and the outside air, convective and short wave radiation heat transfer occur so that the boundary conditions is:

$$-k_1 \frac{\partial T}{\partial x} + h_L T = h_L T_{\infty, L}(t) + q_{\infty, L}(t), \qquad x = 0, \tag{2}$$

where $h_L \left[\mathrm{W \,.\, m^{-2} \,.\, K^{-1}}\right]$ is the surface heat transfer coefficient, $T_{\infty, L}$ is the outside ambient temperature varying according to climatic data and $q_{\infty, L} \left[\mathrm{W \,.\, m^{-2}}\right]$ is the incident short wave radiation flux, corresponding to the solar irradiance for the wavelength between 0.2 μm and 3.0 μm. On the inside interface, the boundary is only submitted to convective exchange with the ambient air $T_{\infty, R}$:

$$k_2 \frac{\partial T}{\partial x} + h_R T = h_R T_{\infty, R}(t), \qquad x = \ell, \tag{3}$$

At the interface between the two materials, the continuity of the heat flux and of the fields is assumed. Thus, two additional equations are formulated:

$$T(x - \varepsilon, t) = T(x + \varepsilon, t), \quad k_1 \frac{\partial T}{\partial x}\bigg|_{x - \varepsilon} = k_2 \frac{\partial T}{\partial x}\bigg|_{x + \varepsilon}, \quad \forall \varepsilon \rightarrow 0, \quad x = \ell_1. \tag{4}$$

Last, initially the wall is in steady state:

$$T\big(x, t = 0\big) = T_0(x), \qquad \forall x \in \Omega_x, \tag{5}$$

where T_0 is a given function of space. Note that for numerical reasons, the mathematical problem is transformed into a dimensionless formulation with scaled quantities.

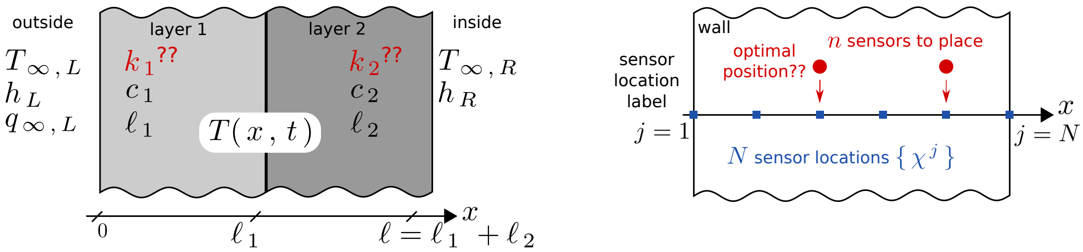
3. Optimal experiment design regarding sensor positioning

A certain number of sensors can be placed in each layer to obtain temperature measurements to solve an inverse problem regarding parameters k_1 and k_2. The issue is to determine the optimal sensor positions for each layer. The methodology to search for an optimal experiment design is now presented.

3.1. Experiment Design

The total number of sensor locations in the wall is $N = N_1 + N_2$, N_1 being in the first layer and N_2 in the second. The set of all possible labels identifying sensor positions is defined by:

$$J = \left\{1, \ldots, N\right\}. \tag{6}$$



(a) (b)

Figure 1 : *Illustration of the multi-layer domain* (a) *and of the problem of optimal experimental design in terms of sensors locations* (b). *For illustration,* $J = \{1, 2, 3, 4, 5, 6\}$ *and* $\xi = \{3, 5\}$.

By extension, we have $J = J_1 \cup J_2$ and $J_1 \cap J_2 = \emptyset$ with $J_1 = \{1, \ldots, N_1\}$ and $J_2 = \{N_1 + 1, \ldots, N_1 + N_2\}$. The set of candidate locations is:

$$\Omega_\chi = \{\chi^j\}_{j \in J}, \tag{7}$$

where $\chi^j\ [\mathrm{m}]$ is a sensor location, so $\chi^j \in \Omega_x$. By extension, the set of sensor location is the union of ones in both layers $\Omega_\chi = \{\chi^j\}_{j \in J_1} \cup \{\chi^j\}_{j \in J_2}$.

The experimental design is illustrated in Figure 1(b). It consists in positioning $n = n_1 + n_2$ sensors ($n \leqslant N$) in the wall at the position labels ξ. So, the experimental design is formulated by:

$$\mathcal{D} = \{\xi, n\}, \qquad n = \mathrm{card}\,\xi, \qquad \xi \subset J. \tag{8}$$

By extension for each layer $i \in \{1, 2\}$, we have $\mathcal{D} = \mathcal{D}_1 \cup \mathcal{D}_2$ with $\mathcal{D}_i = \{\xi_i, n_i\}$, $n_i = \mathrm{card}\,\xi_i$, $\xi_i \subset J_i$.

3.2. Optimal Experiment Design

The optimal experiment design $\mathcal{D}^\circ$ corresponds to the situation where the accuracy of the estimates is maximal. Following the methodology described in [2], the OED is defined by:

$$\mathcal{D}^\circ = \arg\max_{\mathcal{D}} \Phi, \tag{9}$$

with Φ being the D-optimum design criterion:

$$\Phi = \log\det \mathbf{M}, \tag{10}$$

where $\mathbf{M}$ is the so-called FISHER information matrix:

$$\mathbf{M} = \left[M_{pq}\right], \qquad M_{pq} = \sum_{k=1}^{N} w^k \int_0^1 \theta_p(x = \chi^k, t) \cdot \theta_q(x = \chi^k, t)\,\mathrm{d}t, \tag{11}$$

where $\{w^k\}_{k \in J}$ is the set of binary decision variables (also called design weights) indicating whether or not sensor are placed at the locations labeled by the elements of J:

$$w^j = \begin{cases} 1, & j \in \xi, \\ 0, & j \notin \xi, \end{cases} \qquad \forall j \in J. \tag{12}$$

Last, the sensitivity coefficients are:

$$\theta_i = \frac{\sigma_k}{\sigma_T} \frac{\partial T}{\partial k_i}, \quad \forall i \in \{1, 2\}, \tag{13}$$

which are computed by differentiating directly the governing equations (1) with respect to (5) to the corresponding parameters. The constants σ_k and σ_T are set to obtain dimensionless quantities. Complementary works investigating the OED for parameter estimation of transfer phenomena in building porous materials can be consulted in [1, 3, 4].

3.3. Searching for the OED

Three possible strategies to determine the OED are investigated. For better understandings, note that the experimental design can also be defined through the sensor position:

$$\mathcal{D} \equiv \left\{ \{\chi^j\}_{j \in \xi}, n \right\}. \tag{14}$$

In what follows, the designs (*i.e.*, the sets of position labels selected for sensor placement) are understood as the appropriate subsets of J.

3.4. Strategy 1: optimization considering integer parameters

The first one is to consider Eq. (9) as an optimization problem with respect to the N design weights w^j:

$$\mathcal{D} \equiv \{w^j\}_{j \in J}, \tag{15}$$

under the constraint of the maximum number of sensors:

$$\sum_{j=1}^{N} w^j = n, \tag{16}$$

and limiting each design weight to be binary:

$$w^j \in \{0, 1\}, \qquad \forall j \in J. \tag{17}$$

Such problem is solved using the genetic algorithm in the `Matlab` environment with unknown parameters set as integers.

3.5. Strategy 2: optimization of binary design weights using the exchange algorithm

The second strategy also consists in solving the problem as defined in Section 3.4. with binary decision elements. However, an exchange algorithm (1) is used [5]. It runs as follows over the iterations k.

Step 1. At $k = 0$, an initial design is selected:

$$\xi^0 = \xi_1^0 \cup \xi_2^0, \tag{18}$$

where $n = \text{card}\, \xi^0$, $n_1 = \text{card}\, \xi_1^0$ and $n_2 = \text{card}\, \xi_2^0$. For such design, the D-optimum criterion $\Phi\Big(\boldsymbol{M}(\xi^{(0)})\Big)$ is computed using Eqs. (10) and (11).

Step 2. The second step consists in exchanging the position labels of the current design with ones that correspond to vacant sites so as to maximally improve the D-optimum criteria. It is performed by determining the labels $\left(i^\star, j^\star\right)$ such that:

$$\left(i^\star, j^\star\right) = \arg \max_{(i,j) \in S^{(k)}} \Delta\left(i, j\right), \tag{19}$$

where

$$S^{(k)} = \xi^{(k)} \times \left(J \setminus \xi^{(k)}\right), \tag{20}$$

so that $S^{(k)}$ contains all possible exchanges of points, at which a sensor currently resides by points which are currently vacant. The quantity $\Delta\left(i, j\right)$ evaluates the relative changes in the D-optimum criterion:

$$\Delta\left(i, j\right) = \left(\Phi\left(\boldsymbol{M}\left(\xi_{i \leftrightarrow j}\right)\right) - \Phi\left(\boldsymbol{M}\left(\xi^{(k)}\right)\right)\right) \cdot \left(\Phi\left(\boldsymbol{M}\left(\xi^{(k)}\right)\right)\right)^{-1}, \tag{21}$$

where $\xi_{i \leftrightarrow j}$ means the design in which label position i has been replaced by label j.

Step 3. If the relative increase in the D-optimum criterion is lower than a set tolerance

$$\Delta\left(i^\star, j^\star\right) \leqslant \eta, \tag{22}$$

then the algorithm stops since $\xi^{(k)}$ is a locally optimal design. Otherwise, the iterations continues by setting $\xi^{(k+1)} \leftarrow \xi^{(k)}$ and $k \leftarrow k + 1$ and coming back to Step 2. The Algorithm 1 synthesizes the procedure.

Algorithm 1 Exchange algorithm to determine the OED using Strategy 2.

Sample candidate design ξ^0 ▷ *Step 1*
Compute D-optimum criteria $\Phi\left(\boldsymbol{M}\left(\xi^{(0)}\right)\right)$
$k = 0$
while $\Delta\left(i^\star, j^\star\right) \geqslant \eta$ **do**
    State S^k ▷ *Step 2*
    Determine labels $\left(i^\star, j^\star\right)$ according to Eq. (19)
    Compute $\Delta\left(i^\star, j^\star\right)$ with Eq. (21)
    $k = k + 1$
end
Set OED $\mathcal{D}^\circ = \left\{\xi^{k-1}, n\right\}$

3.6. Strategy 3: optimization via convex relaxation

The last strategy is very similar to the first one described in Section 3.4. at the exception that the decision elements are relaxed to be any real numbers in the unit interval [6]:

$$w^j \in \left[0, 1\right], \qquad \forall j \in J. \tag{23}$$

The constraint imposes that the sum of decision elements equals the number of sensors:

$$\sum_{j=1}^{N} w^j = n. \tag{24}$$

As a result, a convenient convex optimization problem is obtained. Then, with determined optimal relaxed weights, the probability distribution of each position sensor can be assessed by:

$$\mathcal{P}^{j} = w^{j} \cdot \left(\sum_{j=1}^{N} w^{j} \right)^{-1} . \tag{25}$$

Such problem is solved using the interior point algorithm in the `Matlab` environment.

4. Case study

The case study considers an outward layer composed of $\ell_1 = 30 \text{ cm}$ stones with a volumetric capacity $c_1 = 2.5 \text{ MJ.m}^{-3}.\text{K}^{-1}$ and an *a priori* thermal conductivity $k_1 = 2.5 \text{ W.m}^{-1}.\text{K}^{-1}$. The inward layer is a $\ell_2 = 20 \text{ cm}$ insulation material with $c_2 = 0.05 \text{ MJ.m}^{-3}.\text{K}^{-1}$ and $k_2 = 0.05 \text{ W.m}^{-1}.\text{K}^{-1}$. The boundary conditions correspond to measurements obtained from a real building monitored from January 8^{th} to April 29^{th} 2009 [1, 7] and illustrated in Figure 2. The initial condition is defined by assuming the wall in a steady state so that:

$$T_0(x) = \begin{cases} 1.21 \cdot x - 0.9, & \forall x \in [0, \ell_1] \\ 60.6 \cdot x - 18.7, & \forall x \in [\ell_1, \ell_2] \end{cases} \quad [°\text{C}]. \tag{26}$$

The simulation horizon is $t_f = 111 \text{ d}$. Regarding the experimental design, $N_1 = 14$ and $N_2 = 9$ candidate sensor positions are possible in layers 1 and 2, respectively. Such values are obtained by constraining a minimum gap of 2 cm between two neighboring sensors and avoiding sensors at the interfaces with inside/outside air or between materials. The three above-described strategies are considered to determine the optimal sensor positions. Tolerances of optimization solvers are set to 10^{-8}. In addition, an exhaustive search is carried out by computing the D-optimum criteria among the $\binom{N}{n}$ possible sensor positions in the wall.

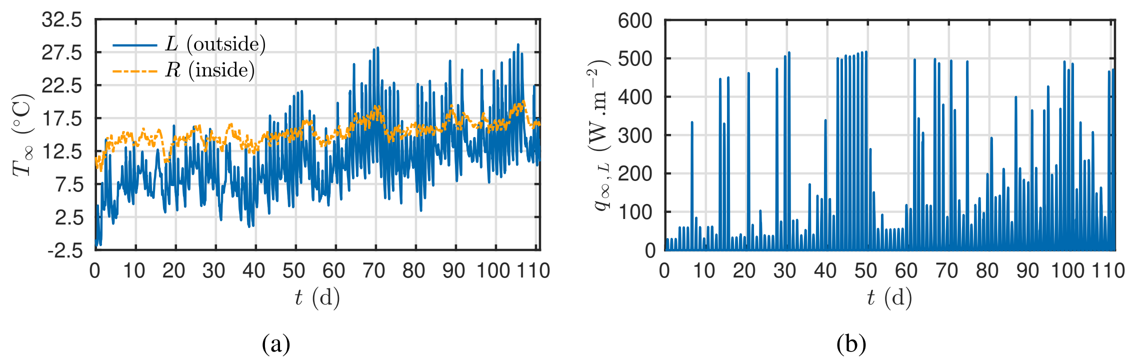


Figure 2 : *Time variation of the boundary conditions for temperature* (a) *and incident heat flux* (b).

Results of all approaches are presented in Table 1 for $n = 5$ sensor to be placed. Recalling that the number of sensor is set before the optimization procedure. For all methods, the initial guess are the same label position, chosen randomly. Note that tests have been performed for different sets of initial guesses (randomly chosen) to verify the consistency of the results. The results are consistent among all strategies. For Strategy 1 and 2, the optimal experimental design is the same with the reference one given by the exhaustive search. For Strategy 3 (real decision elements) results differs for one sensor position. However, the probability density function of the sensor location (Figure 3(a)) indicates a positioning of four sensors consistent with the other approaches. For the fifth sensor, the probability is almost equivalent for a sensor placed at $36\ \mathrm{cm}$ or $42\ \mathrm{cm}$. The first choice is consistent with reference results. Thus, the Strategy 3 gives complementary information since an experimenter could choose among two almost equivalent experiment designs.

To illustrate this last point, Figure 3(b) shows the variation of the sensor positioning probability according to the number of sensors. Starting with $n = 2$, the optimal design consists in placing one sensor in each layer. Two equivalent positions are possible for the sensor in layer 2. With n increasing, there are more sensors to be placed in layer 2. The sensor positioning is consistent among the designs, *i.e.* the sensors are located in the last third of layer 1 and in the middle of layer 2. Those results depends on the sensitivity functions of the problem which are related to the characteristic diffusion time of the materials.

Figure 4(a) shows the experimental design determined using the second strategy with the exchange algorithm according to the iteration number (with $n = 5$ sensors to place). At each iteration, only one sensor position is changed as set in the algorithm. Very few iterations are required for the algorithm to determine a local optimal solution. Compared to other solutions, it is the one with the smallest cost function evaluations. The ratio compared to the exhaustive search scales with $0.01\ \%$. The optimization considering integer decision elements w^j is the strategy with the highest ratio of more than $17\ \%$. In our numerical experiments, this strategy is relatively unstable converging to one of the several local maxima in the discrete space illustrated in Figure 4(b).

	Sensor position $[\mathrm{cm}]$					*Cost function evaluation*	
	χ^1	χ^2	χ^3	χ^4	χ^5	*Number*	*Ratio* $[\%]$
Exhaustive search	26	28	36	38	40	33 649	–
Strategy 1 (int. param., gen. alg.)	26	28	36	38	40	5806	17.25
Strategy 2 (int. param., exch. alg.)	26	28	36	38	40	5	0.01
Strategy 3 (real param., int. point alg.)	26	28	38	40	42	552	1.64

Table 1 : *Optimal Experiment design for $n = 5$ sensor positions. The vertical line designates the interface between two layers.*

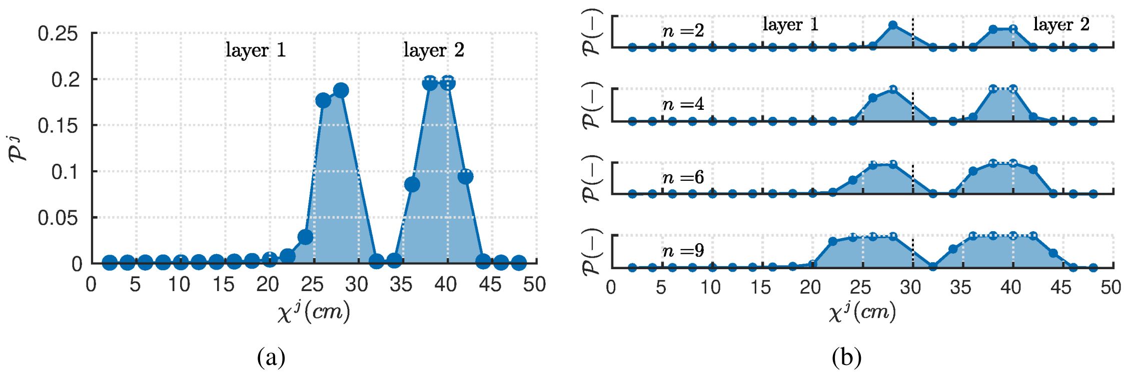
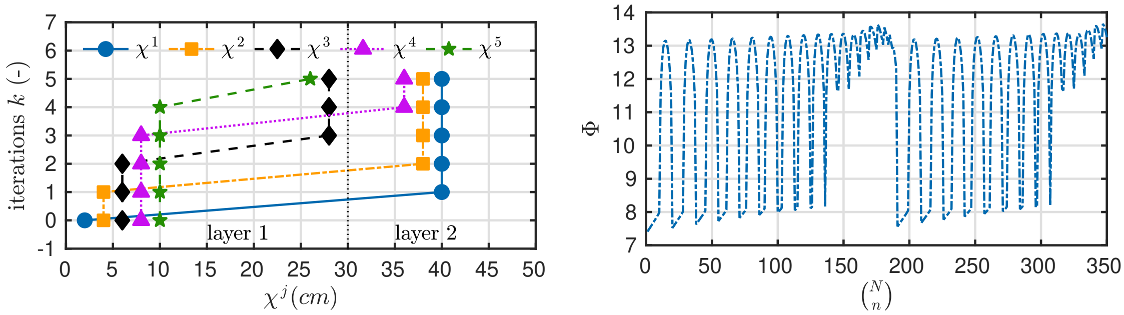


Figure 3 : *Variation of the selected design according to the iterations of the exchange algorithm (Strategy 3)* (a). *Variation of the probability of the sensor position considering real decision elements (Stategy 3) according to the number of sensors* (b).



(a) (b)

Figure 4 : *Probability of the sensor position considering real decision elements (Strategy 2)* (a). *Variation of the D–optimum criteria for* 350 *positions out of the possible combination (*$N = 23$ *and* $n = 5$*)* (b).

5. Conclusion

This article investigates the OED in terms of sensor positioning in a two-layers slab for a thermal conductivity inverse heat conduction problem. The issue is to maximize the D–optimum criterion according to the sensor location. Three algorithms are evaluated to solve the optimization problem. The first one uses a genetic algorithm considering the decision elements to place a sensor at one position as binary ($0/1$ sensor). The second one employs an exchange algorithm. For the last strategy, the problem is relaxed by considering the decision elements as real numbers in the unit interval. In this way, probabilities of sensor positioning are obtained. Results highlight a good consistency of the OED determined by the three approaches. Strategy 2 is the most efficient from a computational point of view while strategy 3 give complementary probabilistic information for the experimenter. Future works should focus on extending the methodology for two-dimensional heat transfer where exhaustive OED search has a too high computational cost requiring alternative strategies. Then, experiments should be performed to determine the unknown parameters.

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