Introduction à l’homogénéisation périodique: milieu à deux phases et résistance thermique interfaciale.

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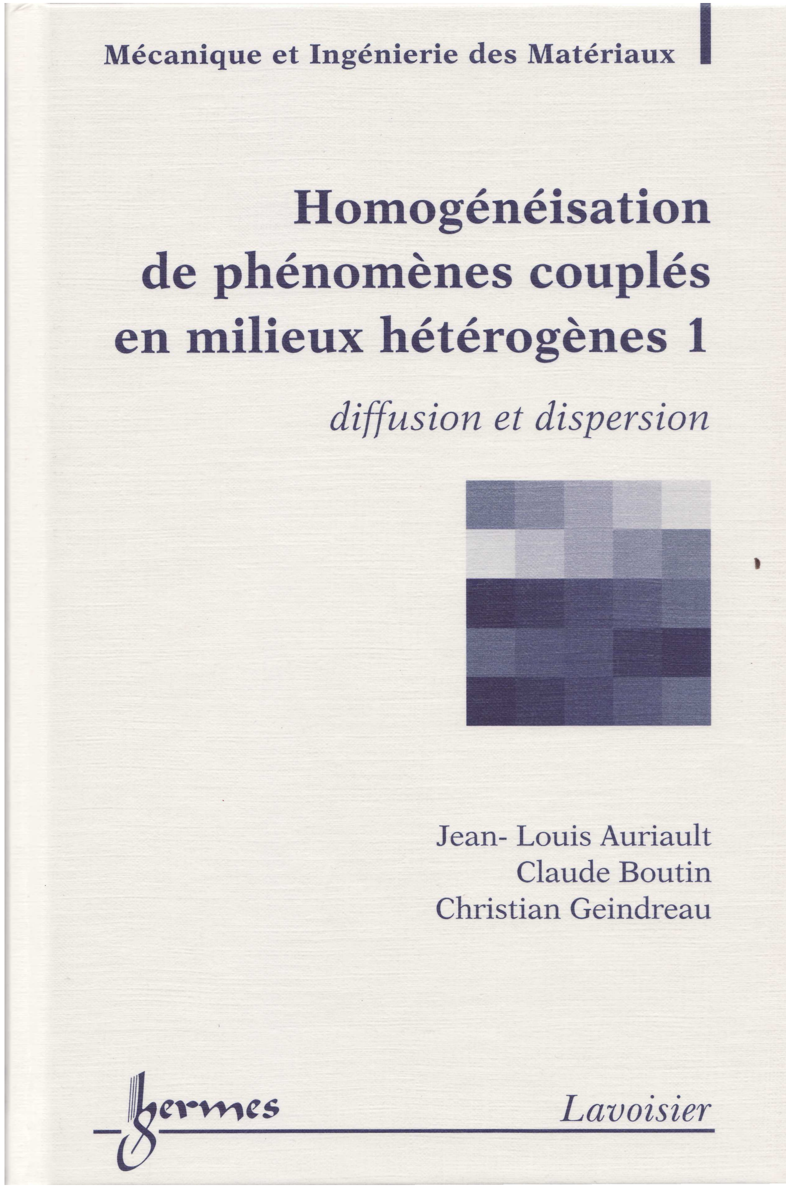
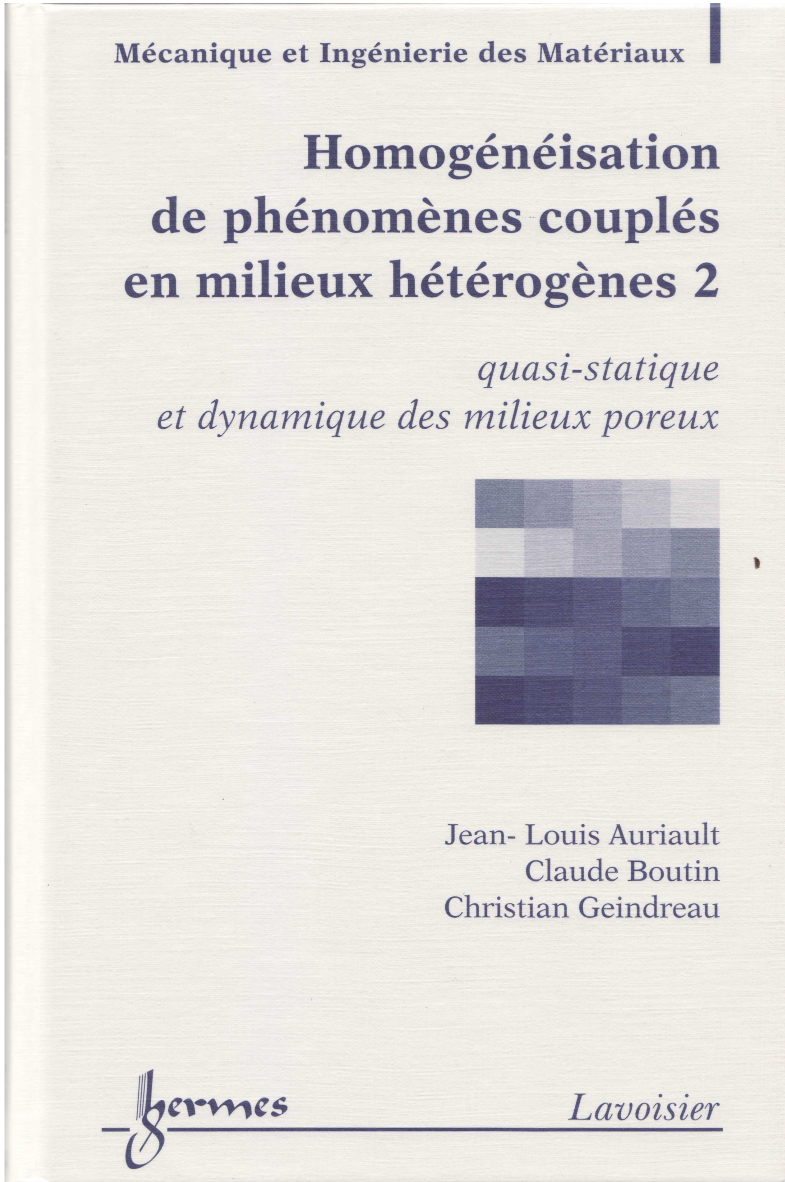
Vandoeuvre lès Nancy

Objectifs de la présentation

- **Introduire la méthode de l'homogénéisation périodique à deux échelles.**
- **Monter qu'elle n'est pas seulement une *méthode mathématique* mais plutôt un *outil de modélisation*.**
- **Pratiquer une pédagogie par l'exemple à partir d'un exemple simple et parlant.**

Suggestions de lecture





Pour les gens pressés…

- **Jean-Louis Auriault and Horia I. Ene**

Macroscopic modelling of heat transfer in composites with interfacial thermal barrier. *Int. J. Heat Mass Transfer*, Vol. 37, N°18, pp. 2885-2892, 1994.

- **Jean-Louis Auriault**

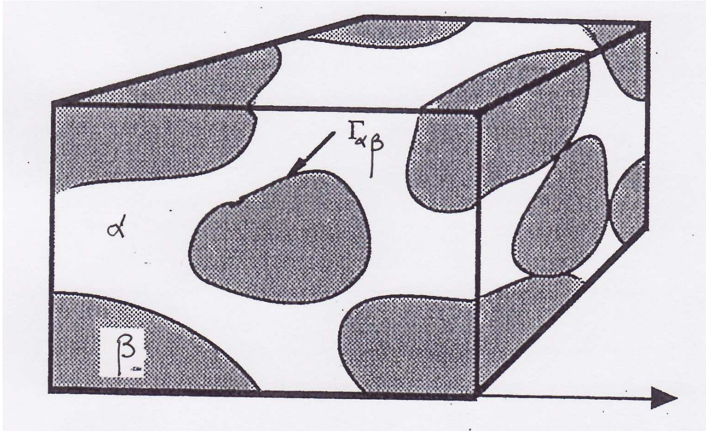
Heterogeneous media: is an equivalent homogeneous description always possible?

***Int. J. Engrg Sci*, Vol. 29, pp. 785-795, 1991.**

Le problème à résoudre

- **Equations dans V_α et V_β**

$$\left\{\begin{array}{lcll} V_\alpha & : & (\rho c)_\alpha \dfrac{\partial T_\alpha}{\partial t} & = & \nabla\cdot(\lambda_\alpha \nabla T_\alpha) \\ V_\beta & : & (\rho c)_\beta \dfrac{\partial T_\beta}{\partial t} & = & \nabla\cdot(\lambda_\beta \nabla T_\beta) \end{array}\right.$$



- **Interface $\Gamma_{\alpha\beta}$**

$$\left\{\begin{array}{lcl} -\lambda_\alpha \nabla T_\alpha \cdot \boldsymbol{n}_\alpha & = & -\lambda_\beta \nabla T_\beta \cdot \boldsymbol{n}_\alpha \\ -\lambda_\alpha \nabla T_\alpha \cdot \boldsymbol{n}_\alpha & = & h\,(T_\alpha - T_\beta) \end{array}\right.$$

- **Conditions aux limites sur les frontières**
- **Condition initiale**

Géométrie du problème

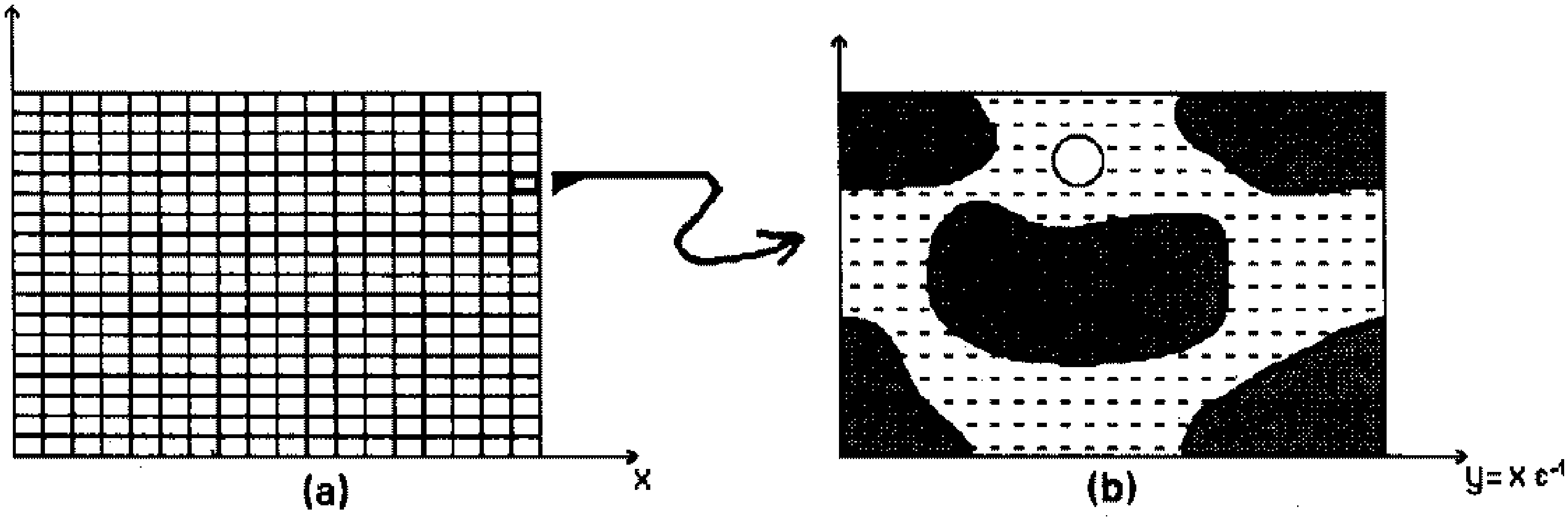


FIG. 1. (a) Macroscopic scale. (b) Microscopic scale, period Ω.

- **Milieu périodique à deux échelles spatiales**
  - ℓ **microscopique : taille de la cellule-unité, du VER, . . .**
  - L **macroscopique : en régime permanent, longueur macroscopique du milieu**
  - $\epsilon = \dfrac{\ell}{L} \ll 1$ **est un petit paramètre**

Méthode de l'homogénéisation périodique (échelles multiples)

- **Mise sous forme adimensionnelle des équations**

  - **longueur caractéristique à choisir librement entre ℓ et L :**

    $\boxed{\boxed{\ell}}$ **et** $\boldsymbol{\nabla}^* = \ell\, \boldsymbol{\nabla}$

  - $\mathcal{O}((\rho c)_\alpha) = \mathcal{O}((\rho c)_\beta)$ **et** $\mathcal{O}(\lambda_\alpha) = \mathcal{O}(\lambda_\beta)$

  - **temps caractéristique τ : $t^* = t/\tau$ avec $\tau = L^2/a_\alpha$ et $a = \lambda/(\rho c)$**

$$
\left\{
\begin{array}{lccll}
V_\alpha & : & \epsilon^2 & \dfrac{\partial T_\alpha}{\partial t^*} & = & \nabla^{*2}\, T_\alpha \\
V_\beta & : & \epsilon^2 & \dfrac{a_\alpha}{a_\beta}\dfrac{\partial T_\beta}{\partial t^*} & = & \nabla^{*2}\, T_\beta \\
\Gamma_{\alpha\beta} & : & & -\boldsymbol{\nabla}^* T_\alpha \cdot \boldsymbol{n}_\alpha & = & -\dfrac{\lambda_\beta}{\lambda_\alpha}\boldsymbol{\nabla}^* T_\beta \cdot \boldsymbol{n}_\alpha \\
 & & & -\boldsymbol{\nabla}^* T_\alpha \cdot \boldsymbol{n}_\alpha & = & \mathrm{Bi}\,(T_\alpha - T_\beta) \quad \textbf{avec} \quad \mathrm{Bi} = \dfrac{h\,\ell}{\lambda_\alpha}
\end{array}
\right.
$$

• Séparabilité des échelles

- **Si $\epsilon = \ell/L \ll 1$, les échelles sont dites séparables,**
- **deux coordonnées d'espace sont nécessaires pour repérer un point :**

$$\left\{\begin{array}{lcll} \boldsymbol{x} & : & \text{coordonnée macroscopique} & \boldsymbol{x}^* = \boldsymbol{x}/L \\ \boldsymbol{y} & : & \text{coordonnée microscopique} & \boldsymbol{y}^* = \boldsymbol{y}/\ell \end{array}\right.$$

- **Séparabilité $\Rightarrow$ $\boldsymbol{x}$ et $\boldsymbol{y}$ sont des variables indépendantes.**
- **Dérivée des fonctions composées :** $\dfrac{\mathrm{d}}{\mathrm{d}\theta}\psi\big(u(\theta), v(\theta)\big) = \dfrac{\partial\psi}{\partial u}\dfrac{\mathrm{du}}{\mathrm{d}\theta} + \dfrac{\partial\psi}{\partial v}\dfrac{\mathrm{dv}}{\mathrm{d}\theta}$

$$\begin{array}{ccccc} \boldsymbol{\nabla} & = & \boldsymbol{\nabla}_x & + & \boldsymbol{\nabla}_y \\ \dfrac{1}{\ell}\boldsymbol{\nabla}^* & = & \dfrac{1}{L}\boldsymbol{\nabla}_{x^*} & + & \dfrac{1}{\ell}\boldsymbol{\nabla}_{y^*} \\ \boldsymbol{\nabla}^* & = & \boldsymbol{\nabla}_{y^*} & + & \epsilon\,\boldsymbol{\nabla}_{x^*} \end{array}$$

Récapitulatif

• **On cherche : $\boldsymbol{T_\alpha(t^*, x^*, y^*)}$ et $\boldsymbol{T_\beta(t^*, x^*, y^*)}$ solutions de :**

$$
\left\{
\begin{array}{llll}
V_\alpha & : & \epsilon^2 \, \dfrac{\partial T_\alpha}{\partial t^*} & = \nabla^{*2} \, T_\alpha \\
V_\beta & : & \epsilon^2 \, \dfrac{a_\alpha}{a_\beta} \dfrac{\partial T_\beta}{\partial t^*} & = \nabla^{*2} \, T_\beta \\
\Gamma_{\alpha\beta} & : & -\boldsymbol{\nabla}^* T_\alpha \cdot \boldsymbol{n}_\alpha & = -\dfrac{\lambda_\beta}{\lambda_\alpha} \boldsymbol{\nabla}^* T_\beta \cdot \boldsymbol{n}_\alpha \\
 & & -\boldsymbol{\nabla}^* T_\alpha \cdot \boldsymbol{n}_\alpha & = \mathrm{Bi} \, (T_\alpha - T_\beta) \quad \textbf{avec} \quad \mathrm{Bi} = \dfrac{h \, \ell}{\lambda_\alpha}
\end{array}
\right.
$$

avec $\boldsymbol{\nabla}^* = \boldsymbol{\nabla}_{y^*} + \epsilon \, \boldsymbol{\nabla}_{x^*}$

• Ecriture pratique du problème

$$
\left\{
\begin{array}{llll}
V_\alpha & : \quad \epsilon^2\,(\rho c)_\alpha \dfrac{\partial T_\alpha}{\partial t} & = & \boldsymbol{\nabla}\cdot(\lambda_\alpha \boldsymbol{\nabla} T_\alpha) \\
V_\beta & : \quad \epsilon^2\,(\rho c)_\beta \dfrac{\partial T_\beta}{\partial t} & = & \boldsymbol{\nabla}\cdot(\lambda_\beta \boldsymbol{\nabla} T_\beta) \\
\Gamma_{\alpha\beta} & : \quad -\lambda_\alpha \boldsymbol{\nabla} T_\alpha \cdot \boldsymbol{n}_\alpha & = & -\lambda_\beta \boldsymbol{\nabla} T_\beta \cdot \boldsymbol{n}_\alpha \\
 & \quad\;\; -\lambda_\alpha \boldsymbol{\nabla} T_\alpha \cdot \boldsymbol{n}_\alpha & = & \epsilon^p\, h\,(T_\alpha - T_\beta)
\end{array}
\right.
$$

avec $\quad \boldsymbol{\nabla} = \boldsymbol{\nabla}_y + \epsilon\,\boldsymbol{\nabla}_x$

• Remarques

– **Les préfacteurs ϵ^q purement formels indiquent l'ordre de grandeur des termes.**

– **Si $\mathcal{O}(\mathrm{Bi}) = \mathcal{O}(\epsilon^p)$, le facteur ϵ^p est devant h.**

– **L'exposant p peut être fixé arbitrairement.**

Mise en œuvre de la méthode

- **Développement en séries de ϵ**

$$T_\alpha = \sum_k T_\alpha^k(t, \boldsymbol{x}, \boldsymbol{y})\epsilon^k \quad \textbf{et} \quad T_\beta = \sum_k T_\beta^k(t, \boldsymbol{x}, \boldsymbol{y})\epsilon^k$$

- **Hiérarchie de problèmes : aux ordres successifs (ϵ^0, ϵ^1, ϵ^2, ...) posés sur la cellule-unité $Y = Y_\alpha \cup Y_\beta$.**

- **Périodicité : $T_{\alpha(\beta)}(t, \boldsymbol{x}, \boldsymbol{y})$ sont périodiques en $\boldsymbol{y}$.**

Conditions de périodicité

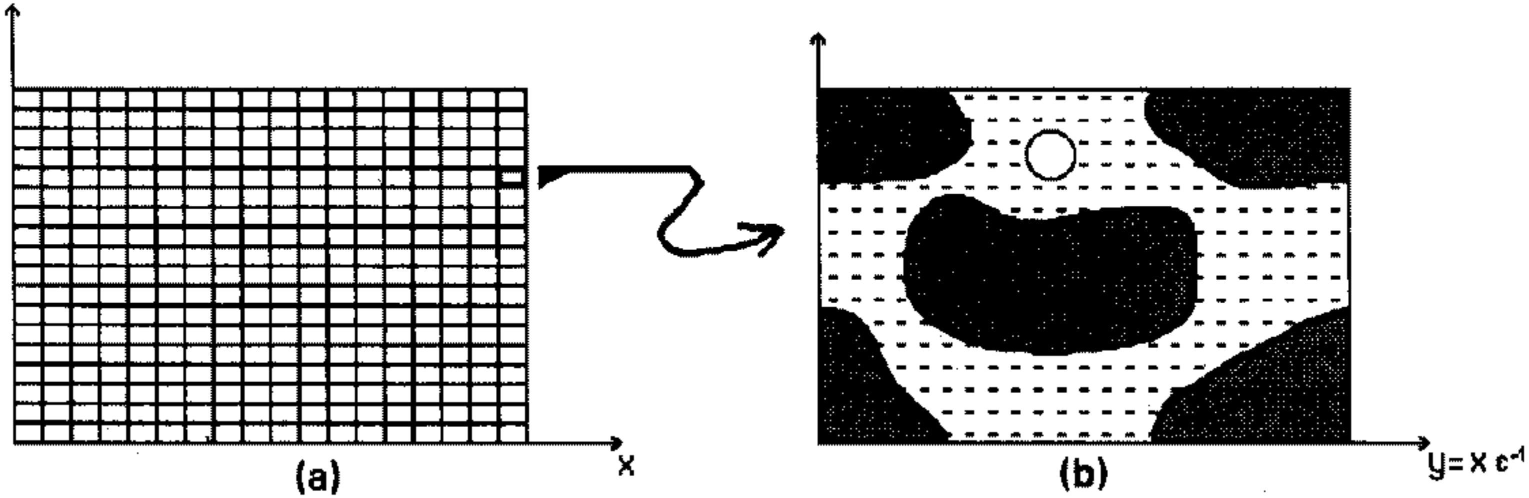


FIG. 1. (a) Macroscopic scale. (b) Microscopic scale, period Ω.

- **Aux points de la frontière** $\partial Y_{\alpha\, ext} \cup \partial Y_{\beta\, ext}$ **se correspondant par périodicité**
  - **Egalité des températures**
  - **Egalité des densités de flux normales**
- **Condition autre que Dirichlet, Neuman ou Fourier souvent implantée dans les codes numériques.**
- **Condition limite sur la cellule plus que périodicité réelle.**

Cas I : très faible résistance interfaciale Bi $= \mathcal{O}(\epsilon^{-1})$

• Problème à résoudre sur la cellule-unité

$$\left\{\begin{array}{llll}
Y_\alpha & : & \epsilon^2\,(\rho c)_\alpha \dfrac{\partial T_\alpha}{\partial t} & = & \boldsymbol{\nabla}\cdot(\lambda_\alpha \boldsymbol{\nabla} T_\alpha) \\
Y_\beta & : & \epsilon^2\,(\rho c)_\beta \dfrac{\partial T_\beta}{\partial t} & = & \boldsymbol{\nabla}\cdot(\lambda_\beta \boldsymbol{\nabla} T_\beta) \\
\partial Y_{\alpha\beta} & : & -\lambda_\alpha \boldsymbol{\nabla} T_\alpha \cdot \boldsymbol{n}_\alpha & = & -\lambda_\beta \boldsymbol{\nabla} T_\beta \cdot \boldsymbol{n}_\alpha \\
 & & T_\alpha - T_\beta & = & \dfrac{\epsilon}{h}\,(-\lambda_\alpha \boldsymbol{\nabla} T_\alpha \cdot \boldsymbol{n}_\alpha)
\end{array}\right.$$

- avec $\boldsymbol{\nabla} = \boldsymbol{\nabla}_y + \epsilon\,\boldsymbol{\nabla}_x$
- Périodicité en $\boldsymbol{y}$
- $T_\alpha = \sum_k T_\alpha^k(\boldsymbol{x},\boldsymbol{y})\epsilon^k$ et $T_\beta = \sum_k T_\beta^k(\boldsymbol{x},\boldsymbol{y})\epsilon^k$

• **Ordre $\mathcal{O}(\epsilon^0)$** : $T^0_{\alpha(\beta)}$ périodique en $\boldsymbol{y}$ (sur $\partial Y_{\alpha\, ext} \cup \partial Y_{\beta\, ext}$)

$$
\left\{
\begin{array}{lcrcl}
Y_\alpha & : & \boldsymbol{\nabla}_y \cdot \left(\lambda_\alpha \boldsymbol{\nabla}_y T^0_\alpha\right) & = & 0 \\
Y_\beta & : & \boldsymbol{\nabla}_y \cdot \left(\lambda_\beta \boldsymbol{\nabla}_y T^0_\beta\right) & = & 0 \\
\partial Y_{\alpha\beta} & : & -\lambda_\alpha \boldsymbol{\nabla}_y T^0_\alpha \cdot \boldsymbol{n}_\alpha & = & -\lambda_\beta \boldsymbol{\nabla}_y T^0_\beta \cdot \boldsymbol{n}_\alpha \\
 & & T^0_\alpha & = & T^0_\beta
\end{array}
\right.
$$

Solution : $T^0_\alpha(t, \boldsymbol{x}, \boldsymbol{y}) = T^0_\beta(t, \boldsymbol{x}, \boldsymbol{y}) \equiv T^0(t, \boldsymbol{x})$

• **Ordre $\mathcal{O}(\epsilon^1)$:** $T^1_{\alpha(\beta)}$ périodique en $\boldsymbol{y}$

$$\left\{\begin{array}{lcrcl} Y_\alpha & : & \boldsymbol{\nabla}_y \cdot \left(\lambda_\alpha \boldsymbol{\nabla}_y T^1_\alpha\right) & = & 0 \\ Y_\beta & : & \boldsymbol{\nabla}_y \cdot \left(\lambda_\beta \boldsymbol{\nabla}_y T^1_\beta\right) & = & 0 \\ \partial Y_{\alpha\beta} & : & -\lambda_\alpha \left(\boldsymbol{\nabla}_y T^1_\alpha + \boldsymbol{\nabla}_x T^0\right) \cdot \boldsymbol{n}_\alpha & = & -\lambda_\beta \left(\boldsymbol{\nabla}_y T^1_\beta + \boldsymbol{\nabla}_x T^0\right) \cdot \boldsymbol{n}_\alpha \\ & & T^1_\alpha & = & T^1_\beta \end{array}\right.$$

Solution : $T^1_{\alpha(\beta)}(t, \boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{\chi}^I_{\alpha(\beta)}(\boldsymbol{y}) \cdot \boldsymbol{\nabla}_x T^0(t, \boldsymbol{x}) + \widehat{T}_{\alpha(\beta)}(t, \boldsymbol{x})$

où le vecteur $\boldsymbol{\chi}^I_{\alpha(\beta)}$ Y-périodique est solution de :

$$\left\{\begin{array}{lcrcl} Y_\alpha & : & \boldsymbol{\nabla}_y \cdot \left(\lambda_\alpha \boldsymbol{\nabla}_y \boldsymbol{\chi}^I_\alpha\right) & = & 0 \\ Y_\beta & : & \boldsymbol{\nabla}_y \cdot \left(\lambda_\beta \boldsymbol{\nabla}_y \boldsymbol{\chi}^I_\beta\right) & = & 0 \\ \partial Y_{\alpha\beta} & : & -\lambda_\alpha \, \boldsymbol{n}_\alpha \cdot \left(\boldsymbol{\nabla}_y \boldsymbol{\chi}^I_\alpha + \boldsymbol{I}\right) & = & -\lambda_\beta \, \boldsymbol{n}_\alpha \cdot \left(\boldsymbol{\nabla}_y \boldsymbol{\chi}^I_\beta + \boldsymbol{I}\right) \\ & & \boldsymbol{\chi}^I_\alpha & = & \boldsymbol{\chi}^I_\beta \end{array}\right.$$

• *Remarque* : Solution pour $\boldsymbol{\chi}$ à une constante (vectorielle) près.

• **Ordre $\mathcal{O}(\epsilon^2)$** : $T^2_{\alpha(\beta)}$ périodique en $\boldsymbol{y}$

$$
\left\{
\begin{array}{lll}
Y_\alpha & : & (\rho c)_\alpha \dfrac{\partial T^0}{\partial t} = \boldsymbol{\nabla}_y \cdot \left[\lambda_\alpha \left(\boldsymbol{\nabla}_y T^2_\alpha + \boldsymbol{\nabla}_x T^1_\alpha\right)\right] + \boldsymbol{\nabla}_x \cdot \left[\lambda_\alpha \left(\boldsymbol{\nabla}_y T^1_\alpha + \boldsymbol{\nabla}_x T^0\right)\right] \\
Y_\beta & : & (\rho c)_\beta \dfrac{\partial T^0}{\partial t} = \boldsymbol{\nabla}_y \cdot \left[\lambda_\beta \left(\boldsymbol{\nabla}_y T^2_\beta + \boldsymbol{\nabla}_x T^1_\beta\right)\right] + \boldsymbol{\nabla}_x \cdot \left[\lambda_\beta \left(\boldsymbol{\nabla}_y T^1_\beta + \boldsymbol{\nabla}_x T^0\right)\right] \\
\partial Y_{\alpha\beta} & : & -\lambda_\alpha \left(\boldsymbol{\nabla}_y T^2_\alpha + \boldsymbol{\nabla}_x T^1_\alpha\right) \cdot \boldsymbol{n}_\alpha = -\lambda_\beta \left(\boldsymbol{\nabla}_y T^2_\beta + \boldsymbol{\nabla}_x T^1_\beta\right) \cdot \boldsymbol{n}_\alpha \\
 & & T^2_\alpha - T^2_\beta = -\dfrac{\lambda_\alpha}{h} \left(\boldsymbol{\nabla}_y T^1_\alpha + \boldsymbol{\nabla}_x T^0\right) \cdot \boldsymbol{n}_\alpha
\end{array}
\right.
$$

Intégration sur Y

$$
\langle \cdot \rangle = \frac{1}{|\mathrm{Y}|} \int_Y \mathrm{dV} = \frac{1}{|\mathrm{Y}|} \left(\int_{\mathrm{Y}_\alpha} \cdot \ \mathrm{dV} + \int_{\mathrm{Y}_\beta} \cdot \ \mathrm{dV} \right)
$$

$$
\begin{aligned}
\langle \rho c \rangle \frac{\partial T^0}{\partial t} &= \frac{1}{|Y|} \int_{\partial Y_{\alpha\beta} \cup \partial Y_{\alpha\, ext}} \lambda_\alpha \left(\boldsymbol{\nabla}_y T_\alpha^2 + \boldsymbol{\nabla}_x T_\alpha^1 \right) \cdot \boldsymbol{n}_\alpha \, \mathrm{dS} \\
&+ \frac{1}{|Y|} \int_{\partial Y_{\alpha\beta} \cup \partial Y_{\beta e}} \lambda_\beta \left(\boldsymbol{\nabla}_y T_\beta^2 + \boldsymbol{\nabla}_x T_\beta^1 \right) \cdot \boldsymbol{n}_\beta \, \mathrm{dS} \\
&+ \boldsymbol{\nabla}_x \cdot \left\langle \lambda \left(\boldsymbol{\nabla}_y \boldsymbol{\chi}^I + \boldsymbol{I} \right) \boldsymbol{\nabla}_x T^0 \right\rangle \\
&= \boldsymbol{\nabla}_x \cdot \left\langle \lambda \left(\boldsymbol{\nabla}_y \boldsymbol{\chi}^I + \boldsymbol{I} \right) \boldsymbol{\nabla}_x T^0 \right\rangle
\end{aligned}
$$

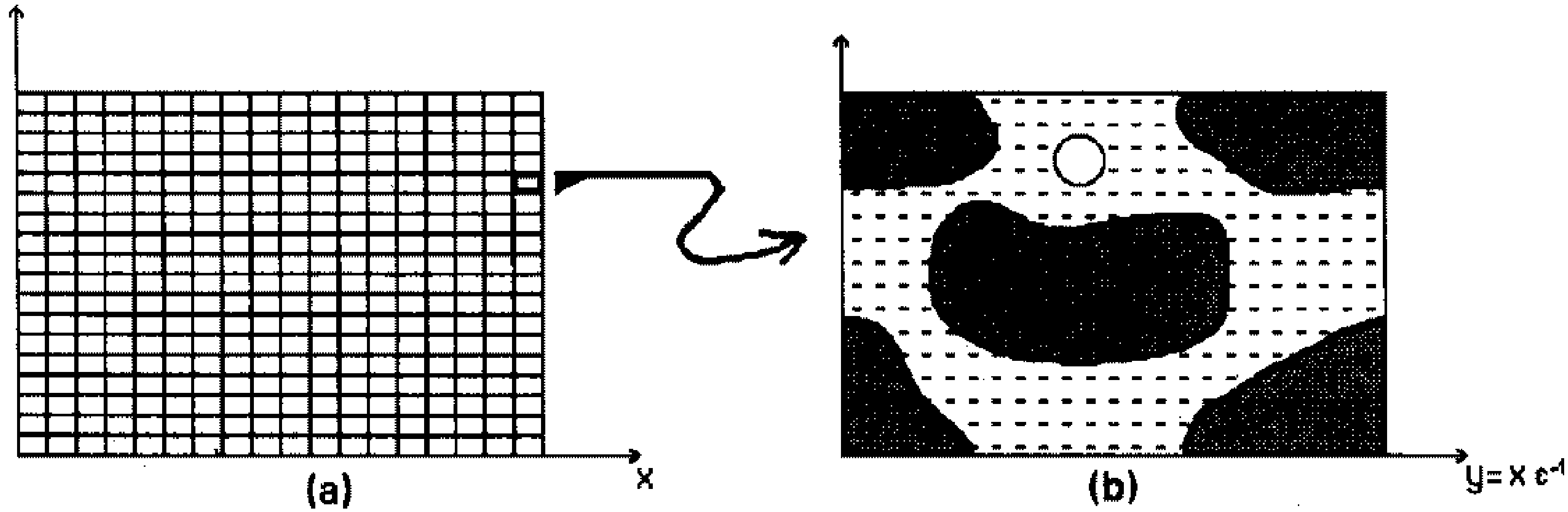


FIG. 1. (a) Macroscopic scale. (b) Microscopic scale, period Ω.

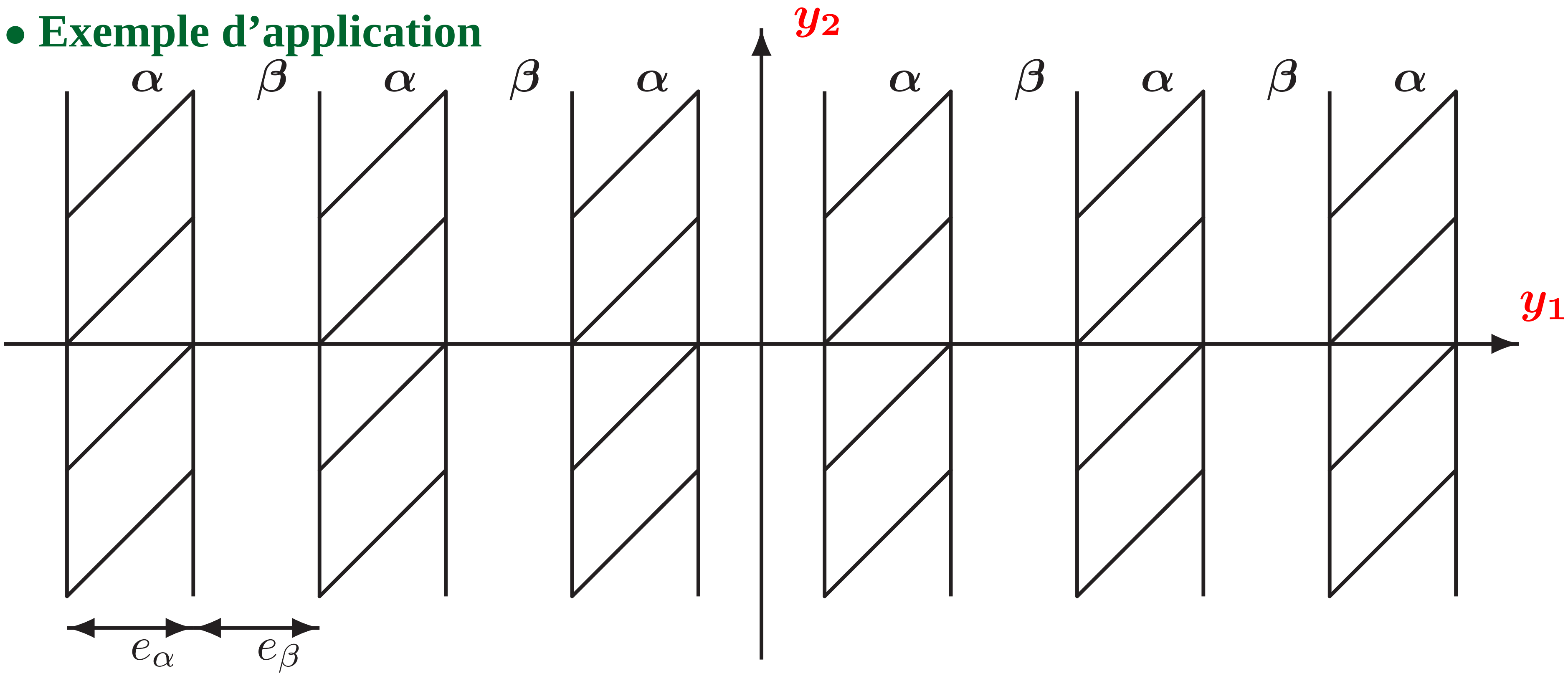
Résultat

$$\boxed{\langle \rho c \rangle \frac{\partial T^0}{\partial t} = \boldsymbol{\nabla}_x \cdot \left(\boldsymbol{\lambda}_{eq}^I \boldsymbol{\nabla}_x T^0 \right)}$$

$$\boldsymbol{\lambda}_{eq}^I = \left\langle \lambda \left(\boldsymbol{\nabla}_y \boldsymbol{\chi}^I + \boldsymbol{I} \right) \right\rangle$$

$$\langle \rho c \rangle = n_\alpha \left(\rho c \right)_\alpha + n_\beta \left(\rho c \right)_\beta$$

- **Exemple d’application**



$$\left\{\begin{array}{lcrcl}
Y_\alpha & : & \boldsymbol{\nabla}_y \cdot \left(\lambda_\alpha \boldsymbol{\nabla}_y \boldsymbol{\chi}^I_\alpha\right) & = & 0 \\
Y_\beta & : & \boldsymbol{\nabla}_y \cdot \left(\lambda_\beta \boldsymbol{\nabla}_y \boldsymbol{\chi}^I_\beta\right) & = & 0 \\
\partial Y_{\alpha\beta} & : & -\lambda_\alpha \boldsymbol{n}_\alpha \cdot \left(\boldsymbol{\nabla}_y \boldsymbol{\chi}^I_\alpha + \boldsymbol{I}\right) & = & -\lambda_\beta \boldsymbol{n}_\alpha \cdot \left(\boldsymbol{\nabla}_y \boldsymbol{\chi}^I_\beta + \boldsymbol{I}\right) \\
 & & \boldsymbol{\chi}^I_\alpha & = & \boldsymbol{\chi}^I_\beta \\
 & & \multicolumn{3}{l}{\boldsymbol{\chi}^I_{\alpha(\beta)} \text{ périodique } en\ \boldsymbol{y}}
\end{array}\right.$$

$$\boldsymbol{\chi} = \begin{pmatrix} \chi|_1 \\ \chi|_2 \end{pmatrix} \qquad \boldsymbol{\chi}_{\alpha(\beta)} \text{ invariant en } \boldsymbol{y}_2$$

dans Y_α : $\lambda_\alpha \dfrac{\mathrm{d}^2\ \chi_\alpha|_1}{\mathrm{dy}_1^2} = 0$ $\qquad \lambda_\alpha \dfrac{\mathrm{d}^2\ \chi_\alpha|_2}{\mathrm{dy}_1^2} = 0$

dans Y_β : $\lambda_\beta \dfrac{\mathrm{d}^2\ \chi_\beta|_1}{\mathrm{dy}_1^2} = 0$ $\qquad \lambda_\beta \dfrac{\mathrm{d}^2\ \chi_\beta|_2}{\mathrm{dy}_1^2} = 0$

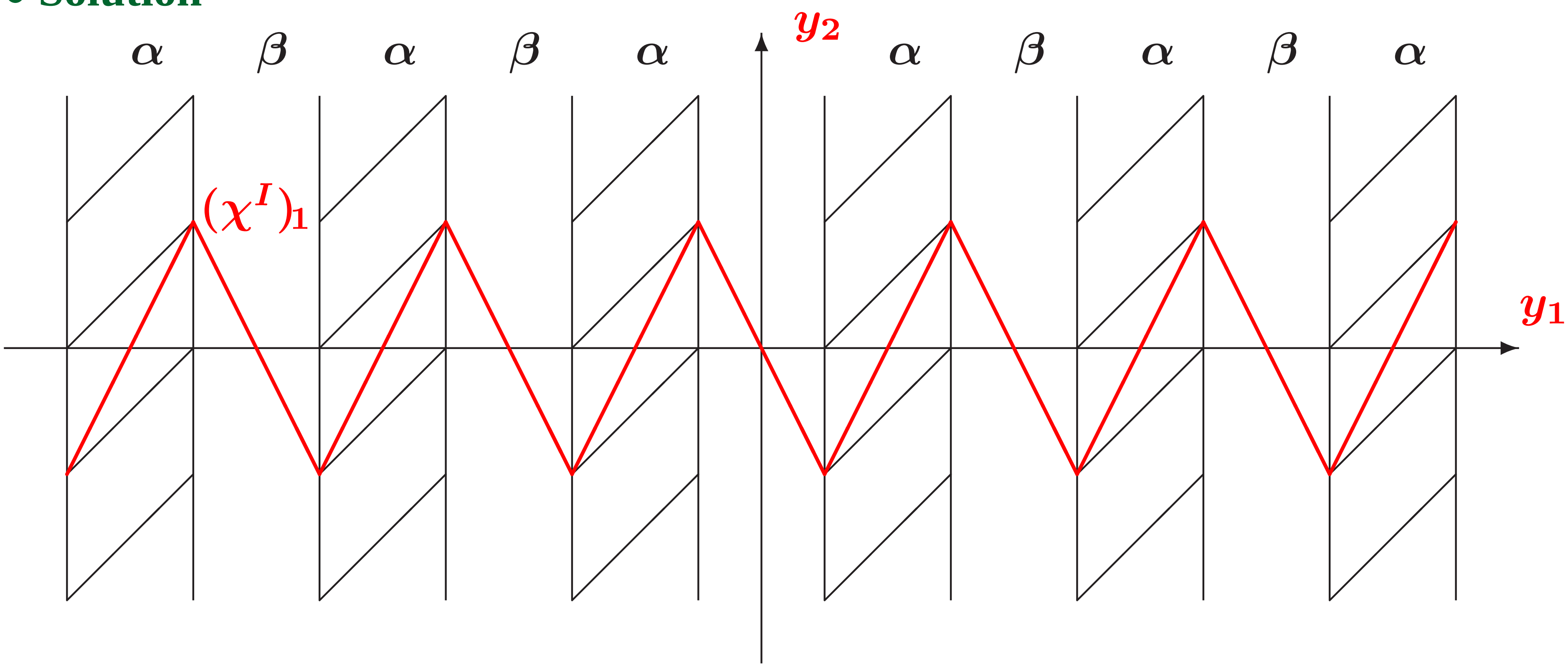
sur $\partial Y_{\alpha\beta}$: $\chi_\alpha^I\big|_1 = \chi_\beta^I\big|_1$ $\qquad \chi_\alpha^I\big|_2 = \chi_\beta^I\big|_2$

$$-\lambda_\alpha \left(\frac{\mathrm{d}\ \chi_\alpha^{\mathrm{I}}\big|_1}{\mathrm{dy}_1} + 1 \right) = -\lambda_\beta \left(\frac{\mathrm{d}\ \chi_\beta^{\mathrm{I}}\big|_1}{\mathrm{dy}_1} + 1 \right) \qquad -\lambda_\alpha \frac{\mathrm{d}\ \chi_\alpha^{\mathrm{I}}\big|_2}{\mathrm{dy}_1} = -\lambda_\beta \frac{\mathrm{d}\ \chi_\beta^{\mathrm{I}}\big|_2}{\mathrm{dy}_1}$$

Périodicité $\qquad$ Périodicité

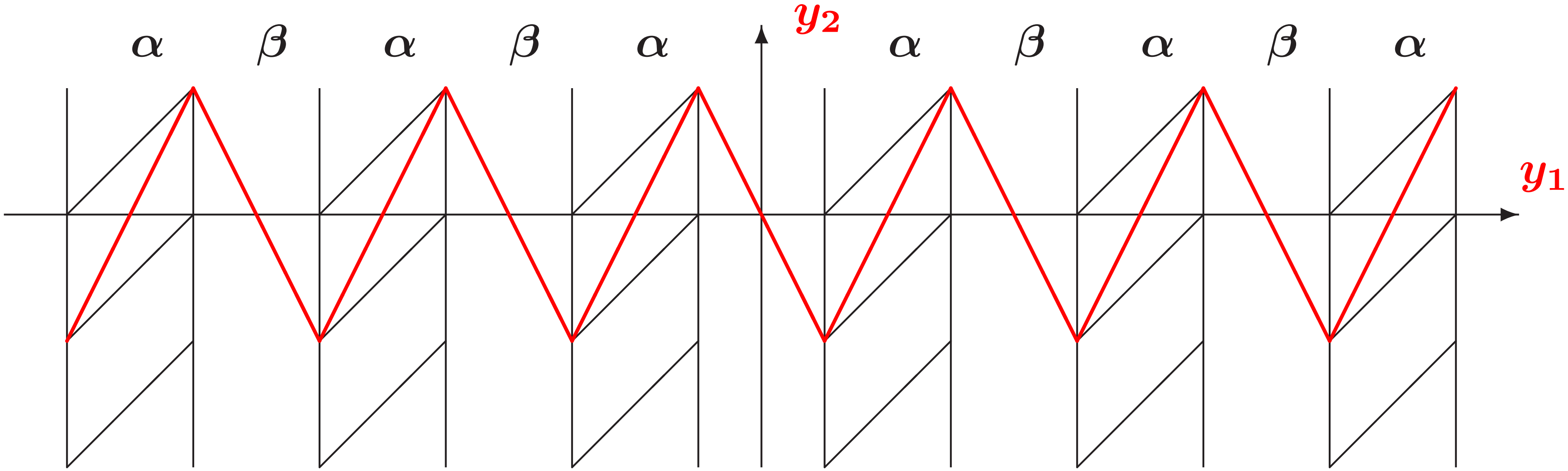
Remarque : $\boldsymbol{\chi}$ défini à une constante près.

- **Solution**



$$(\chi^I)_2 \quad = \quad 0$$

• Résultat



$$\frac{\mathrm{d}(\chi^{\mathrm{I}}_{\alpha})_1}{\mathrm{dy}_1} = \frac{\lambda_\beta - \lambda_\alpha}{\lambda_\alpha + \lambda_\beta \dfrac{e_\alpha}{e_\beta}} \qquad \frac{\mathrm{d}(\chi^{\mathrm{I}}_{\beta})_1}{\mathrm{dy}_1} = \frac{\lambda_\alpha - \lambda_\beta}{\lambda_\alpha \dfrac{e_\beta}{e_\alpha} + \lambda_\beta} \qquad \left(\chi^I_\alpha\right)_2 = 0$$

$$\boldsymbol{\lambda}^I_{eq} = \langle \lambda \left(\boldsymbol{\nabla}_y \boldsymbol{\chi}^I + \boldsymbol{I} \right) \rangle = \begin{pmatrix} \dfrac{1}{\dfrac{n_\alpha}{\lambda_\alpha} + \dfrac{n_\beta}{\lambda_\beta}} & 0 \\ 0 & n_\alpha \, \lambda_\alpha + n_\beta \, \lambda_\beta \end{pmatrix} \quad \text{avec} \quad n_{\alpha(\beta)} = \frac{e_{\alpha(\beta)}}{e_\alpha + e_\beta}$$

Cas II : résistance de contact modérée Bi $= \mathcal{O}(\epsilon^0)$

- **Problème à résoudre**

$$
\left\{
\begin{array}{lclcl}
Y_\alpha & : & \epsilon^2 \, (\rho c)_\alpha \dfrac{\partial T_\alpha}{\partial t} & = & \boldsymbol{\nabla} \cdot (\lambda_\alpha \boldsymbol{\nabla} T_\alpha) \\
Y_\beta & : & \epsilon^2 \, (\rho c)_\beta \dfrac{\partial T_\beta}{\partial t} & = & \boldsymbol{\nabla} \cdot (\lambda_\beta \boldsymbol{\nabla} T_\beta) \\
\partial Y_{\alpha\beta} & : & -\lambda_\alpha \boldsymbol{\nabla} T_\alpha \cdot \boldsymbol{n}_\alpha & = & -\lambda_\beta \boldsymbol{\nabla} T_\beta \cdot \boldsymbol{n}_\alpha \\
 & & T_\alpha - T_\beta & = & \dfrac{1}{h} \left(-\lambda_\alpha \boldsymbol{\nabla} T_\alpha \cdot \boldsymbol{n}_\alpha \right)
\end{array}
\right.
$$

avec $\qquad \boldsymbol{\nabla} = \boldsymbol{\nabla}_y + \epsilon \, \boldsymbol{\nabla}_x$

• **Ordre $\mathcal{O}(\epsilon^0)$** : $T^0_{\alpha(\beta)}$ périodique en $\boldsymbol{y}$

$$
\left\{
\begin{array}{lcrcl}
Y_\alpha & : & \boldsymbol{\nabla}_y \cdot \left(\lambda_\alpha \boldsymbol{\nabla}_y T^0_\alpha\right) & = & 0 \\
Y_\beta & : & \boldsymbol{\nabla}_y \cdot \left(\lambda_\beta \boldsymbol{\nabla}_y T^0_\beta\right) & = & 0 \\
\partial Y_{\alpha\beta} & : & -\lambda_\alpha \boldsymbol{\nabla}_y T^0_\alpha \cdot \boldsymbol{n}_\alpha & = & -\lambda_\beta \boldsymbol{\nabla}_y T^0_\beta \cdot \boldsymbol{n}_\alpha \\
 & & T^0_\alpha - T^0_\beta & = & \dfrac{1}{h}\left(-\lambda_\alpha \boldsymbol{\nabla}_y T^0_\alpha \cdot \boldsymbol{n}_\alpha\right)
\end{array}
\right.
$$

Solution : $T^0_\alpha(t, \boldsymbol{x}, \boldsymbol{y}) = T^0_\beta(t, \boldsymbol{x}, \boldsymbol{y}) \equiv T^0(t, \boldsymbol{x})$

• **Ordre $\mathcal{O}(\epsilon^1)$** : $T^1_{\alpha(\beta)}$ périodique en $\boldsymbol{y}$

$$\left\{\begin{array}{lcrcl} Y_\alpha & : & \boldsymbol{\nabla}_y \cdot \left(\lambda_\alpha \boldsymbol{\nabla}_y T^1_\alpha\right) & = & 0 \\ Y_\beta & : & \boldsymbol{\nabla}_y \cdot \left(\lambda_\beta \boldsymbol{\nabla}_y T^1_\beta\right) & = & 0 \\ \partial Y_{\alpha\beta} & : & -\lambda_\alpha \left(\boldsymbol{\nabla}_y T^1_\alpha + \boldsymbol{\nabla}_x T^0\right) \cdot \boldsymbol{n}_\alpha & = & -\lambda_\beta \left(\boldsymbol{\nabla}_y T^1_\beta + \boldsymbol{\nabla}_x T^0\right) \cdot \boldsymbol{n}_\alpha \\ & & T^1_\alpha - T^1_\beta & = & -\dfrac{\lambda_\alpha}{h} \left(\boldsymbol{\nabla}_y T^1_\alpha + \boldsymbol{\nabla}_x T^0\right) \cdot \boldsymbol{n}_\alpha \end{array}\right.$$

Solution : $T^1_{\alpha(\beta)}(t, \boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{\chi}^{II}_{\alpha(\beta)}(\boldsymbol{y}) \cdot \boldsymbol{\nabla}_x T^0(t, \boldsymbol{x}) + \widehat{T}_{\alpha(\beta)}(t, \boldsymbol{x})$

où le vecteur $\boldsymbol{\chi}^{II}_{\alpha(\beta)}$ Y-périodique est solution de :

$$\left\{\begin{array}{lcrcl} Y_\alpha & : & \boldsymbol{\nabla}_y \cdot \left(\lambda_\alpha \boldsymbol{\nabla}_y \boldsymbol{\chi}^{II}_\alpha\right) & = & 0 \\ Y_\beta & : & \boldsymbol{\nabla}_y \cdot \left(\lambda_\beta \boldsymbol{\nabla}_y \boldsymbol{\chi}^{II}_\beta\right) & = & 0 \\ \partial Y_{\alpha\beta} & : & -\lambda_\alpha \boldsymbol{n}_\alpha \cdot \left(\boldsymbol{\nabla}_y \boldsymbol{\chi}^{II}_\alpha + \boldsymbol{I}\right) & = & -\lambda_\beta\, \boldsymbol{n}_\alpha \cdot \left(\boldsymbol{\nabla}_y \boldsymbol{\chi}^{II}_\beta + \boldsymbol{I}\right) \\ & & h\left(\boldsymbol{\chi}^{II}_\alpha - \boldsymbol{\chi}^{II}_\beta\right) & = & -\lambda_\alpha\, \boldsymbol{n}_\alpha \cdot \left(\boldsymbol{\nabla}_y \boldsymbol{\chi}^{II}_\alpha + \boldsymbol{I}\right) \end{array}\right.$$

• **Ordre $\mathcal{O}(\epsilon^2)$**: $T^2_{\alpha(\beta)}$ périodique en $\boldsymbol{y}$

$$\left\{\begin{array}{lll} Y_\alpha & : & (\rho c)_\alpha \dfrac{\partial T^0}{\partial t} = \boldsymbol{\nabla}_y \cdot \left[\lambda_\alpha \left(\boldsymbol{\nabla}_y T^2_\alpha + \boldsymbol{\nabla}_x T^1_\alpha\right)\right] + \boldsymbol{\nabla}_x \cdot \left[\lambda_\alpha \left(\boldsymbol{\nabla}_y T^1_\alpha + \boldsymbol{\nabla}_x T^0\right)\right] \\ Y_\beta & : & (\rho c)_\beta \dfrac{\partial T^0}{\partial t} = \boldsymbol{\nabla}_y \cdot \left[\lambda_\beta \left(\boldsymbol{\nabla}_y T^2_\beta + \boldsymbol{\nabla}_x T^1_\beta\right)\right] + \boldsymbol{\nabla}_x \cdot \left[\lambda_\beta \left(\boldsymbol{\nabla}_y T^1_\beta + \boldsymbol{\nabla}_x T^0\right)\right] \\ \partial Y_{\alpha\beta} & : & -\lambda_\alpha \left(\boldsymbol{\nabla}_y T^2_\alpha + \boldsymbol{\nabla}_x T^1_\alpha\right) \cdot \boldsymbol{n}_\alpha = -\lambda_\beta \left(\boldsymbol{\nabla}_y T^2_\beta + \boldsymbol{\nabla}_x T^1_\beta\right) \cdot \boldsymbol{n}_\alpha \\ & & T^2_\alpha - T^2_\beta = -\dfrac{\lambda_\alpha}{h} \left(\boldsymbol{\nabla}_y T^2_\alpha + \boldsymbol{\nabla}_x T^1_\alpha\right) \cdot \boldsymbol{n}_\alpha \end{array}\right.$$

Intégration sur Y

$$\langle \cdot \rangle = \frac{1}{|\mathrm{Y}|} \int_Y \mathrm{dV} = \frac{1}{|\mathrm{Y}|} \left(\int_{\mathrm{Y}_\alpha} \cdot \ \mathrm{dV} + \int_{\mathrm{Y}_\beta} \cdot \ \mathrm{dV} \right)$$

$$
\begin{aligned}
\langle \rho c \rangle \frac{\partial T^0}{\partial t} &= \frac{1}{|Y|} \int_{\partial Y_{\alpha\beta} \cup \partial Y_{\alpha\, ext}} \left(\lambda_\alpha \boldsymbol{\nabla}_y T^2_\alpha + \boldsymbol{\nabla}_x T^1_\alpha \right) \cdot \boldsymbol{n}_\alpha \, \mathrm{dS} \\
&+ \frac{1}{|Y|} \int_{\partial Y_{\alpha\beta} \cup \partial Y_{\beta e}} \left(\lambda_\beta \boldsymbol{\nabla}_y T^2_\beta + \boldsymbol{\nabla}_x T^1_\beta \right) \cdot \boldsymbol{n}_\beta \, \mathrm{dS} \\
&+ \boldsymbol{\nabla}_x \cdot \left\langle \lambda \left(\boldsymbol{\nabla}_y \boldsymbol{\chi}^{II} + \boldsymbol{I} \right) \boldsymbol{\nabla}_x T^0 \right\rangle \\
&= \boldsymbol{\nabla}_x \cdot \left\langle \lambda \left(\boldsymbol{\nabla}_y \boldsymbol{\chi}^{II} + \boldsymbol{I} \right) \boldsymbol{\nabla}_x T^0 \right\rangle
\end{aligned}
$$

Résultat

$$
\boxed{\langle \rho c \rangle \frac{\partial T^0}{\partial t} = \boldsymbol{\nabla}_x \cdot \left(\boldsymbol{\lambda}^{II}_{eq} \boldsymbol{\nabla}_x T^0 \right)}
$$

$$
\boldsymbol{\lambda}^{II}_{eq} = \left\langle \lambda \left(\boldsymbol{\nabla}_y \boldsymbol{\chi}^{II} + \boldsymbol{I} \right) \right\rangle
$$

$$
\langle \rho c \rangle = n_\alpha \left(\rho c \right)_\alpha + n_\beta \left(\rho c \right)_\beta
$$

Cas III : forte résistance de contact Bi $= \mathcal{O}(\epsilon^1)$

- **Problème à résoudre**

$$
\left\{
\begin{array}{lcrcl}
Y_\alpha & : & \epsilon^2 \, (\rho c)_\alpha \dfrac{\partial T_\alpha}{\partial t} & = & \boldsymbol{\nabla} \cdot (\lambda_\alpha \boldsymbol{\nabla} T_\alpha) \\
Y_\beta & : & \epsilon^2 \, (\rho c)_\beta \dfrac{\partial T_\beta}{\partial t} & = & \boldsymbol{\nabla} \cdot (\lambda_\beta \boldsymbol{\nabla} T_\beta) \\
\partial Y_{\alpha\beta} & : & -\lambda_\alpha \boldsymbol{\nabla} T_\alpha \cdot \boldsymbol{n}_\alpha & = & -\lambda_\beta \boldsymbol{\nabla} T_\beta \cdot \boldsymbol{n}_\alpha \\
 & & \epsilon \, h \, (T_\alpha - T_\beta) & = & -\lambda_\alpha \boldsymbol{\nabla} T_\alpha \cdot \boldsymbol{n}_\alpha
\end{array}
\right.
$$

$$\text{avec} \qquad \boldsymbol{\nabla} = \boldsymbol{\nabla}_y + \epsilon \, \boldsymbol{\nabla}_x$$

• **Ordre $\mathcal{O}(\epsilon^0)$** : $T^0_{\alpha(\beta)}$ périodique en $\boldsymbol{y}$

$$\left\{\begin{array}{lccc} Y_\alpha & : & \boldsymbol{\nabla}_y \cdot \left(\lambda_\alpha \boldsymbol{\nabla}_y T^0_\alpha\right) & = & 0 \\ Y_\beta & : & \boldsymbol{\nabla}_y \cdot \left(\lambda_\beta \boldsymbol{\nabla}_y T^0_\beta\right) & = & 0 \\ \partial Y_{\alpha\beta} & : & -\lambda_\alpha \boldsymbol{\nabla}_y T^0_\alpha \cdot \boldsymbol{n}_\alpha & = & -\lambda_\beta \boldsymbol{\nabla}_y T^0_\beta \cdot \boldsymbol{n}_\alpha = 0 \end{array}\right.$$

Solution :

$$\begin{array}{lcl} T^0_\alpha(t, \boldsymbol{x}, \boldsymbol{y}) & = & T^0_\alpha(t, \boldsymbol{x}) \\ T^0_\beta(t, \boldsymbol{x}, \boldsymbol{y}) & = & T^0_\beta(t, \boldsymbol{x}) \end{array}$$

• **Ordre $\mathcal{O}(\epsilon^1)$** : $T^1_{\alpha(\beta)}$ périodique en $\boldsymbol{y}$

$$\left\{\begin{array}{lcrcl} Y_\alpha & : & \boldsymbol{\nabla}_y \cdot \left[\lambda_\alpha \left(\boldsymbol{\nabla}_y T^1_\alpha + \boldsymbol{\nabla}_x T^0_\alpha\right)\right] & = & 0 \\ Y_\beta & : & \boldsymbol{\nabla}_y \cdot \left[\lambda_\beta \left(\boldsymbol{\nabla}_y T^1_\beta + \boldsymbol{\nabla}_x T^0_\beta\right)\right] & = & 0 \\ \partial Y_{\alpha\beta} & : & -\lambda_\alpha \left(\boldsymbol{\nabla}_y T^1_\alpha + \boldsymbol{\nabla}_x T^0_\alpha\right) \cdot \boldsymbol{n}_\alpha & = & -\lambda_\beta \left(\boldsymbol{\nabla}_y T^1_\beta + \boldsymbol{\nabla}_x T^0_\beta\right) \cdot \boldsymbol{n}_\alpha \\ & & T^0_\alpha - T^0_\beta & = & -\dfrac{\lambda_\alpha}{h} \left(\boldsymbol{\nabla}_y T^1_\alpha + \boldsymbol{\nabla}_x T^0_\alpha\right) \cdot \boldsymbol{n}_\alpha \end{array}\right.$$

Intégration de l’équation sur Y_α :

$$\begin{aligned} 0 &= \int_{Y_\alpha} \boldsymbol{\nabla}_y \cdot \left[\lambda_\alpha \left(\boldsymbol{\nabla}_y T^1_\alpha + \boldsymbol{\nabla}_x T^0_\alpha\right)\right] \mathrm{dV} = \int_{\partial \mathrm{Y}_{\alpha\beta} \cup \partial \mathrm{Y}_{\alpha\ \mathrm{ext}}} \lambda_\alpha \left(\boldsymbol{\nabla}_\mathrm{y} \mathrm{T}^1_\alpha + \boldsymbol{\nabla}_\mathrm{x} \mathrm{T}^0_\alpha\right) \cdot \boldsymbol{n}_\alpha \, \mathrm{dS} \\ &= \int_{\partial Y_{\alpha\beta}} \lambda_\alpha \left(\boldsymbol{\nabla}_y T^1_\alpha + \boldsymbol{\nabla}_x T^0_\alpha\right) \cdot \boldsymbol{n}_\alpha \, \mathrm{dS} = h \left(T^0_\beta - T^0_\alpha\right) \int_{\partial Y_{\alpha\beta}} \mathrm{dS} = \mathrm{h} \left(\mathrm{T}^0_\beta - \mathrm{T}^0_\alpha\right) |\partial \mathrm{Y}_{\alpha\beta}| \end{aligned}$$

Conclusion :

$$\begin{aligned} T^0_\alpha(t, \boldsymbol{x}) &= T^0_\beta(t, \boldsymbol{x}) \equiv T^0(t, \boldsymbol{x}) \\ -\lambda_\alpha \left(\boldsymbol{\nabla}_y T^1_\alpha + \boldsymbol{\nabla}_x T^0_\alpha\right) \cdot \boldsymbol{n}_\alpha &= -\lambda_\beta \left(\boldsymbol{\nabla}_y T^1_\beta + \boldsymbol{\nabla}_x T^0_\beta\right) \cdot \boldsymbol{n}_\alpha = 0 \end{aligned}$$

Solution :

Deux phases isolées $\Rightarrow$ Deux problèmes non couplés pour T_α^1 et T_β^1

$$T^1_{\alpha(\beta)}(t,\boldsymbol{x},\boldsymbol{y}) \quad = \quad \boldsymbol{\chi}^{III}_{\alpha(\beta)}(\boldsymbol{y}) \cdot \boldsymbol{\nabla}_x T^0(t,\boldsymbol{x}) + \widehat{T}_{\alpha(\beta)}(t,\boldsymbol{x})$$

où les vecteurs $\boldsymbol{\chi}^{III}_{\alpha(\beta)}$ Y-périodiques sont solutions de :

$$\left\{ \begin{array}{l} Y_\alpha \ : \ \boldsymbol{\nabla}_y \cdot \left(\lambda_\alpha \boldsymbol{\nabla}_y \boldsymbol{\chi}^{III}_\alpha\right) = 0 \\ \partial Y_{\alpha\beta} \ : \ -\lambda_\alpha \boldsymbol{n}_\alpha \cdot \left(\boldsymbol{\nabla}_y \boldsymbol{\chi}^{III}_\alpha + \boldsymbol{I}\right) = 0 \end{array} \right. \qquad \left\{ \begin{array}{l} Y_\beta \ : \ \boldsymbol{\nabla}_y \cdot \left(\lambda_\beta \boldsymbol{\nabla}_y \boldsymbol{\chi}^{III}_\beta\right) = 0 \\ \partial Y_{\alpha\beta} \ : \ -\lambda_\beta \boldsymbol{n}_\beta \cdot \left(\boldsymbol{\nabla}_y \boldsymbol{\chi}^{III}_\beta + \boldsymbol{I}\right) = 0 \end{array} \right.$$

Intégration du problème à l'ordre $\mathcal{O}(\epsilon^2)$ sur Y

Résultat

$$\boxed{\langle \rho c \rangle \frac{\partial T^0}{\partial t} = \boldsymbol{\nabla}_x \cdot \left(\boldsymbol{\lambda}_{eq}^{III} \boldsymbol{\nabla}_x T^0 \right)}$$

$$\begin{aligned}
\boldsymbol{\lambda}_{eq}^{III} &= \left(\boldsymbol{\lambda}_{eq}^{III} \right)_\alpha + \left(\boldsymbol{\lambda}_{eq}^{III} \right)_\beta \\
\left(\boldsymbol{\lambda}_{eq}^{III} \right)_\alpha &= \frac{1}{|Y|} \int_{Y_\alpha} \lambda_\alpha \left(\boldsymbol{\nabla}_y \boldsymbol{\chi}_\alpha^{III} + \boldsymbol{I} \right) \mathrm{dV} \\
\left(\boldsymbol{\lambda}_{eq}^{III} \right)_\beta &= \frac{1}{|Y|} \int_{Y_\beta} \lambda_\beta \left(\boldsymbol{\nabla}_y \boldsymbol{\chi}_\beta^{III} + \boldsymbol{I} \right) \mathrm{dV} \\
\langle \rho c \rangle &= n_\alpha \left(\rho c \right)_\alpha + n_\beta \left(\rho c \right)_\beta
\end{aligned}$$

Cas IV : modèle à deux champs de température avec couplage $\mathbf{Bi} = \mathcal{O}(\epsilon^2)$

- **Problème à résoudre**

$$\left\{\begin{array}{lcrcl} Y_\alpha & : & \epsilon^2\,(\rho c)_\alpha \dfrac{\partial T_\alpha}{\partial t} & = & \boldsymbol{\nabla}\cdot(\lambda_\alpha \boldsymbol{\nabla} T_\alpha) \\ Y_\beta & : & \epsilon^2\,(\rho c)_\beta \dfrac{\partial T_\beta}{\partial t} & = & \boldsymbol{\nabla}\cdot(\lambda_\beta \boldsymbol{\nabla} T_\beta) \\ \partial Y_{\alpha\beta} & : & -\lambda_\alpha \boldsymbol{\nabla} T_\alpha \cdot \boldsymbol{n}_\alpha & = & -\lambda_\beta \boldsymbol{\nabla} T_\beta \cdot \boldsymbol{n}_\alpha \\ & & \epsilon^2\,h\,(T_\alpha - T_\beta) & = & -\lambda_\alpha \boldsymbol{\nabla} T_\alpha \cdot \boldsymbol{n}_\alpha \end{array}\right.$$

$$\text{avec} \qquad \boldsymbol{\nabla} = \boldsymbol{\nabla}_y + \epsilon\,\boldsymbol{\nabla}_x$$

- **Ordre $\mathcal{O}(\epsilon^0)$:** (Cf. cas précédent)

$$\begin{aligned} T^0_\alpha(t,\boldsymbol{x},\boldsymbol{y}) &= T^0_\alpha(t,\boldsymbol{x}) \\ T^0_\beta(t,\boldsymbol{x},\boldsymbol{y}) &= T^0_\beta(t,\boldsymbol{x}) \end{aligned}$$

- **Ordre $\mathcal{O}(\epsilon^1)$** : $T^1_{\alpha(\beta)}$ périodique en $\boldsymbol{y}$

$$\left\{ \begin{array}{lcrcl} Y_\alpha & : & \boldsymbol{\nabla}_y \cdot \left[\lambda_\alpha \left(\boldsymbol{\nabla}_y T^1_\alpha + \boldsymbol{\nabla}_x T^0_\alpha\right)\right] & = & 0 \\ Y_\beta & : & \boldsymbol{\nabla}_y \cdot \left[\lambda_\beta \left(\boldsymbol{\nabla}_y T^1_\beta + \boldsymbol{\nabla}_x T^0_\beta\right)\right] & = & 0 \\ \partial Y_{\alpha\beta} & : & -\lambda_\alpha \left(\boldsymbol{\nabla}_y T^1_\alpha + \boldsymbol{\nabla}_x T^0_\alpha\right) \cdot \boldsymbol{n}_\alpha & = & -\lambda_\beta \left(\boldsymbol{\nabla}_y T^1_\beta + \boldsymbol{\nabla}_x T^0_\beta\right) \cdot \boldsymbol{n}_\alpha \\ & & -\lambda_\alpha \left(\boldsymbol{\nabla}_y T^1_\alpha + \boldsymbol{\nabla}_x T^0_\alpha\right) \cdot \boldsymbol{n}_\alpha & = & 0 \end{array} \right.$$

Solution

- Problème analogue au cas précédent $\boldsymbol{\chi}^{IV} \equiv \boldsymbol{\chi}^{III}$
- Mais $T^0_\alpha(t,\boldsymbol{x}) \neq T^0_\beta(t,\boldsymbol{x})$

• **Ordre $\mathcal{O}(\epsilon^2)$:** $T^2_{\alpha(\beta)}$ périodique en $\boldsymbol{y}$

$$
\left\{
\begin{array}{lll}
Y_\alpha & : & (\rho c)_\alpha \dfrac{\partial T^0_\alpha}{\partial t} = \boldsymbol{\nabla}_y \cdot \left[\lambda_\alpha \left(\boldsymbol{\nabla}_y T^2_\alpha + \boldsymbol{\nabla}_x T^1_\alpha\right)\right] + \boldsymbol{\nabla}_x \cdot \left[\lambda_\alpha \left(\boldsymbol{\nabla}_y T^1_\alpha + \boldsymbol{\nabla}_x T^0_\alpha\right)\right] \\
Y_\beta & : & (\rho c)_\beta \dfrac{\partial T^0_\beta}{\partial t} = \boldsymbol{\nabla}_y \cdot \left[\lambda_\beta \left(\boldsymbol{\nabla}_y T^2_\beta + \boldsymbol{\nabla}_x T^1_\beta\right)\right] + \boldsymbol{\nabla}_x \cdot \left[\lambda_\beta \left(\boldsymbol{\nabla}_y T^1_\beta + \boldsymbol{\nabla}_x T^0_\beta\right)\right] \\
\partial Y_{\alpha\beta} & : & -\lambda_\alpha \left(\boldsymbol{\nabla}_y T^2_\alpha + \boldsymbol{\nabla}_x T^1_\alpha\right) \cdot \boldsymbol{n}_\alpha = -\lambda_\beta \left(\boldsymbol{\nabla}_y T^2_\beta + \boldsymbol{\nabla}_x T^1_\beta\right) \cdot \boldsymbol{n}_\alpha \\
 & & h\left(T^0_\alpha - T^0_\beta\right) = -\lambda_\alpha \left(\boldsymbol{\nabla}_y T^2_\alpha + \boldsymbol{\nabla}_x T^1_\alpha\right) \cdot \boldsymbol{n}_\alpha
\end{array}
\right.
$$

Intégration sur $\boldsymbol{Y_\alpha}$ et $\boldsymbol{Y_\beta}$

Résultat : modèle à deux équations

$$\left\{ \begin{array}{lcl} n_\alpha \left(\rho c\right)_\alpha \dfrac{\partial T_\alpha^0}{\partial t} & = & \boldsymbol{\nabla}_x \cdot \left[\left(\boldsymbol{\lambda}_{eq}^{IV} \right)_\alpha \boldsymbol{\nabla}_x T_\alpha^0 \right] + H \left(T_\beta^0 - T_\alpha^0 \right) \\ n_\beta \left(\rho c\right)_\beta \dfrac{\partial T_\beta^0}{\partial t} & = & \boldsymbol{\nabla}_x \cdot \left[\left(\boldsymbol{\lambda}_{eq}^{IV} \right)_\beta \boldsymbol{\nabla}_x T_\beta^0 \right] - H \left(T_\beta^0 - T_\alpha^0 \right) \end{array} \right.$$

$$\left(\boldsymbol{\lambda}_{eq}^{IV} \right)_\alpha = \left(\boldsymbol{\lambda}_{eq}^{III} \right)_\alpha \qquad \left(\boldsymbol{\lambda}_{eq}^{IV} \right)_\beta = \left(\boldsymbol{\lambda}_{eq}^{III} \right)_\beta \qquad H = \frac{\left| \partial Y_{\alpha\beta} \right|}{\left| Y \right|} h$$

Cas V : modèle à deux champs de température sans couplage $\mathrm{Bi} = \mathcal{O}(\epsilon^3)$

La solution est immédiate.

Modèle à deux équations sans échange

$$\left\{ \begin{array}{rcl} n_\alpha \left(\rho c\right)_\alpha \dfrac{\partial T_\alpha^0}{\partial t} & = & \boldsymbol{\nabla}_x \cdot \left[\left(\boldsymbol{\lambda}_{eq}^V \right)_\alpha \boldsymbol{\nabla}_x T_\alpha^0 \right] \\ n_\beta \left(\rho c\right)_\beta \dfrac{\partial T_\beta^0}{\partial t} & = & \boldsymbol{\nabla}_x \cdot \left[\left(\boldsymbol{\lambda}_{eq}^V \right)_\beta \boldsymbol{\nabla}_x T_\beta^0 \right] \end{array} \right.$$

$$\left(\boldsymbol{\lambda}_{eq}^V\right)_\alpha = \left(\boldsymbol{\lambda}_{eq}^{IV}\right)_\alpha = \left(\boldsymbol{\lambda}_{eq}^{III}\right)_\alpha \qquad \left(\boldsymbol{\lambda}_{eq}^V\right)_\beta = \left(\boldsymbol{\lambda}_{eq}^{IV}\right)_\beta = \left(\boldsymbol{\lambda}_{eq}^{III}\right)_\beta$$

Récapitulation

Bi $= h\,\ell/\lambda_1$		ϵ^{-1}	ϵ^{0}	ϵ^{1}	ϵ^{2}	ϵ^{3}	
Modèle	$\Leftarrow$	I	II	III	IV	V	$\Rightarrow$
Equation(s)		1	1	1	2	2	

- **Modèle à une équation le plus puissant**

- **Modèle à deux équations le plus puissant**