Réduction de modèle

Application aux écoulements décollés

Bérengère Podvin
EM2C, Centralesupélec, Université Paris-Saclay

Journée SFT: Apport des modèles réduits dans la modélisation des transferts thermiques

Context

Machine Learning

Digital Twin

Representation

Complex physics

Model order reduction

High-dimensional observations

Physics-informed/ guided/aware models

Data Science

Data assimilation

POD reduced-order models

1. Data-driven representation: empirical Karhunen-Loève basis
2. Physics-based model: Galerkin projection + closure
3. Application to separated flows
4. Summary

The Karhunen-Loève theorem

- Let $u(\eta)$ be a centered stochastic process defined in $L^2(\Omega)$: $E[u] = 0$
- η is a deterministic variable in a closed bounded set Ω (compact set)
- Let us define the covariance function $K(\eta, \eta') = E[u(\eta)u(\eta')]$
- Mercer theorem: There is a denumerable set of eigenvalues $\lambda_n \geq 0$ and a set of eigenvectors $e_n(\eta)$ such that

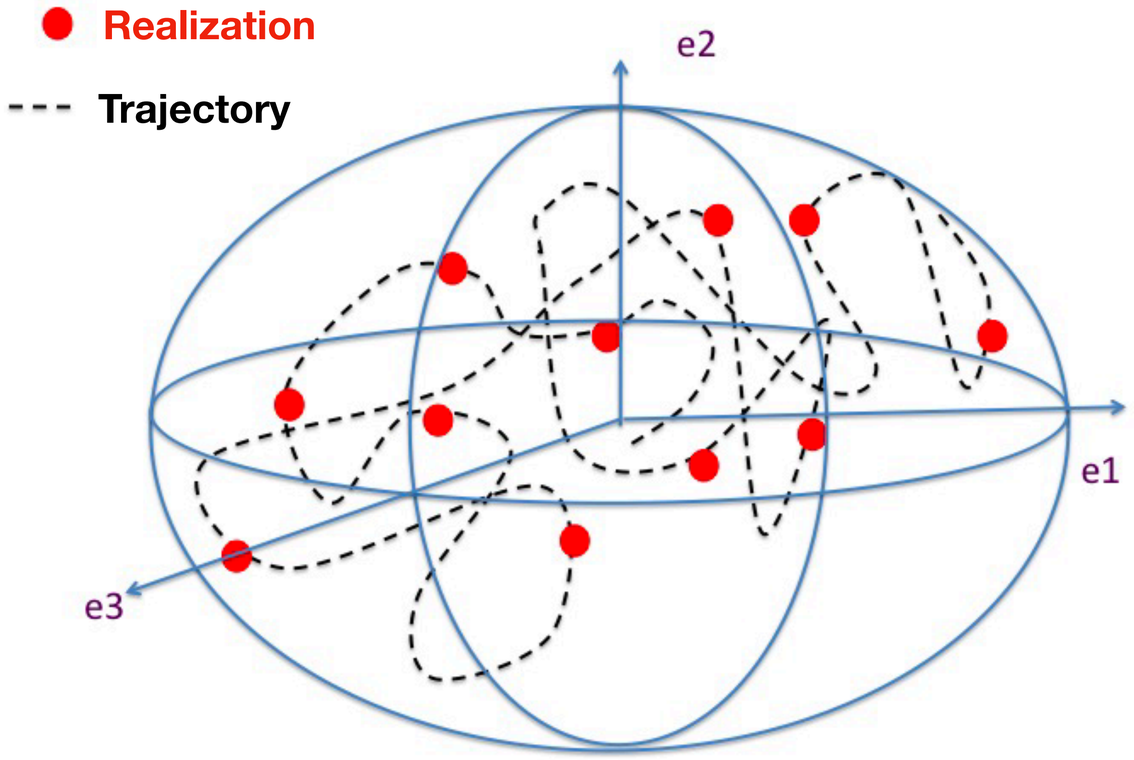
$$K(\eta, \eta') = \sum_{n=1}^{\infty} \lambda_n e_n(\eta) e_n(\eta')$$

Computing the covariance

- Usual choice for a spatio-temporal field (classical POD):
    - Deterministic variables: physical space
    - Stochastic variables: time
- Relies on **ergodicity assumption**: computing the sampled covariance along one trajectory provides a good estimate of the covariance

$$K(\eta, \eta') = \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} u(\eta, t_n) u(\eta', t_n)$$

Interpretation of the empirical modes

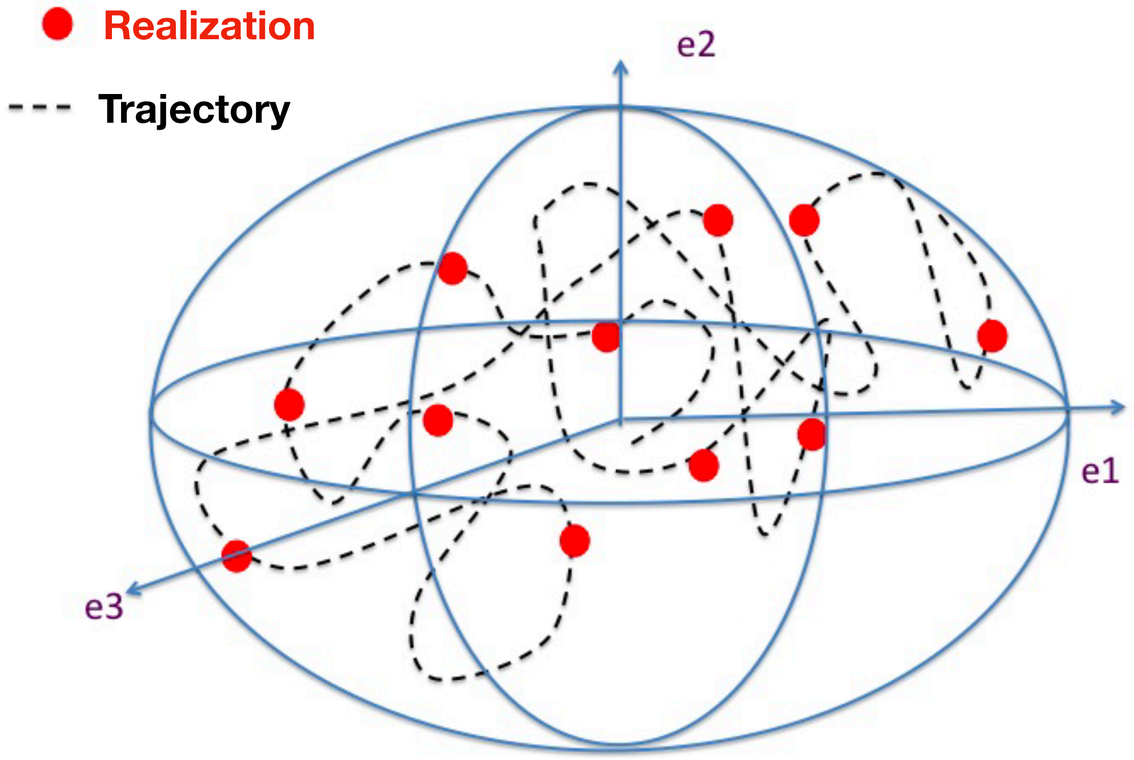


The POD modes are the principal axes of the ellipsoid containing the realizations

$\Longrightarrow$ POD of sampled covariance = PCA

PCA: Principal Component Analysis

Interpretation of the empirical modes



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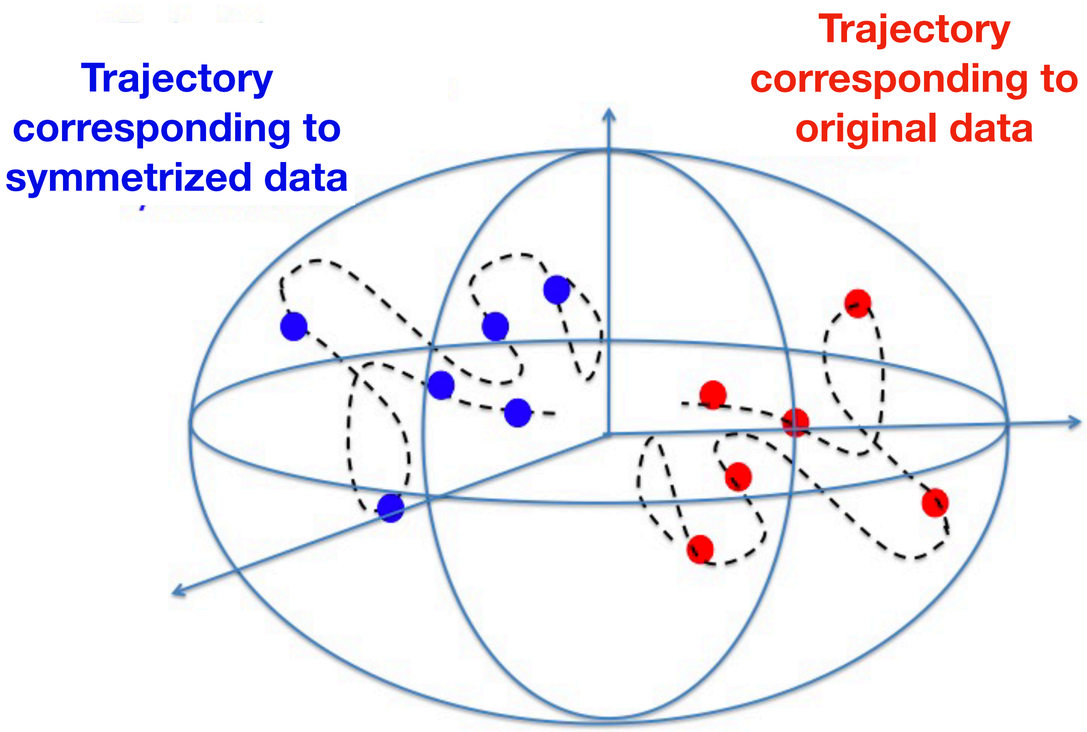
PCA: Principal Component Analysis

Should symmetry be enforced in the dataset?



Are we interested in:

- A single trajectory?
- All possible trajectories (lack of ergodicity)?

POD modes are **not** coherent structures

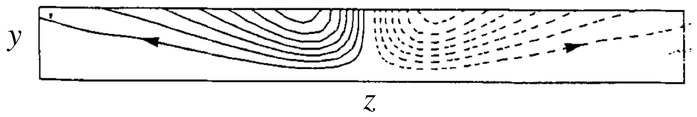
Coherent structures are **localized** in space and time.

POD modes have a **global** support and **integrate** statistical symmetries of the flow.

→POD modes do not generally coincide with coherent structures which can make physical interpretation difficult.

- Example: Cross-section of the turbulent wall layer

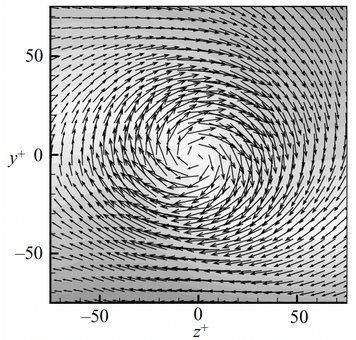
Dominant POD mode (Moin et al. 84)



Symmetric

Invariance by spanwise reflection

Conditional eddy structure (Carlier et al. 05)



Asymmetric

Building a low-dimensional model

Data

$q(x_j, t_k)$

POD

$$q(x_j, t_k) = \sum_n a_n(t_k)\phi_n(x_j)$$

$\{\phi_n(x_j)\}$ +

$\{a_n(t)\}$

$a_n(t_k)$ from POD

or $a_n(t) = \int q(x,t)\phi_n(x)dx$

Conservation equation (PDE)

$\partial_t q + q\nabla q = \nu\nabla^2 q + f(q)(E)$

Galerkin

$(E \mid \phi_n)$

Dynamical system:

$\dot{a}_n = L_{nm}a_m + Q_{nmp}a_m a_p + T_n$

$\rightarrow a_n^{model}(t)$

Application to the incompressible Navier-Stokes equations

1 - Decompose velocity field $\tilde{u} = U + u$ with $U = \bar{\tilde{u}}$

2- Write conservation equations: $\nabla . u = 0$

$$\frac{\partial u}{\partial t} + U . \nabla u + u . \nabla U + u . \nabla u = -\frac{1}{\rho} \nabla p + \nu \nabla^2 u \; (E)$$

+ Boundary conditions

3- Apply POD to fluctuating velocity field u and obtain basis $\{\phi_n\}$

4- Mass conservation is naturally enforced by construction $\forall n, \nabla . \phi_n = 0$ (linear property)

5- Introduce expansion $u(x,t) = \sum_{n=1}^{N} a_n(t)\phi_n(x)$ and project (E) onto mode ϕ_k

Linear terms

- $$\int \frac{\partial u}{\partial t} \,.\, \phi_p(x)dx = \sum_{n=1}^{N} \dot{a}_n(t) \int \phi_n(x)\phi_p(x)dx = \sum_{n=1}^{N} \dot{a}_n(t)\delta_{np} = \dot{a}_p$$

- $$\int \nabla^2 u \,.\, \phi_p(x)dx = \sum_{n=1}^{N} a_n(t) \int \nabla^2\phi_n(x)\phi_p(x)dx = \sum_{n=1}^{N} L^D_{np}a_n(t) \quad L^D \sim \text{diagonal with entries } <0$$

- $$\int (U \,.\, \nabla u + u \,.\, \nabla U) \,.\, \phi_p(x)dx = \sum_{n=1}^{N} a_n(t) \int (U \,.\, \nabla\phi_n(x) + \phi_n(x) \,.\, \nabla U) \,.\, \phi_p(x)dx = \sum_{n=1}^{N} L^M_{np}a_n(t)$$

$L^M \sim \geq 0$ large scales extract energy from mean shear

The Reynolds stress tensor

- $\tau_{ij} = u_i u_j = u_{i,<}u_{j,<} + u_{i,>}u_{j,<} + u_{i,<}u_{j,>} + u_{i,>}u_{j,>}$

RESOLVED CROSS-TERMS UNRESOLVED

- $u_{i,<}u_{j,<}$: computed with Galerkin procedure (quadratic terms) ✓
- $u_{i,<}u_{j,>} + u_{i,>}u_{j,<}$: Leonard stresses can be neglected ✓
- $u_{i,>}u_{j,>}$: modelled with eddy viscosity hypothesis $\tau_> = 2\mu_T s_<$ with $s_< = \frac{1}{2}\left(\nabla u_< + \nabla u_<^T\right)$

The eddy viscosity hypothesis

Heisenberg 1948 « The way in which smaller eddies transfer momentum is similar to ordinary friction »

$$\Longrightarrow \quad \nabla . \tau_{>} = \nu_T \nabla^2 u_{<} \quad \text{with } \nu_T \propto u_C l_C$$

Aubry et al. 1988

$\nu_T = \nu_0$

Extra terms $L^{\nu}_{np} a_n$

$L^{\nu}_{np} = \alpha L^{D}_{np}$

Rempfer and Fasel 1994

$\nu_T = \nu_T(p)$

Extra terms $L^{\nu}_{np} a_n$

$L^{\nu}_{np} = \alpha_p L^{D}_{np}$

Osth et al. 2014

$\nu_T \propto k_{<}^{1/2}$

Extra terms $D^{\nu}_{p} \left(\sum_n |a_n^2| \right)^{1/2} a_p$

$k_{<} = \sum_{n=1}^{N} |a_n|^2$

Podvin and Sergent 2017

$$\nu_T \propto k_{<}^{1/2} \approx \overline{k}_{<}^{1/2} \left(1 + \frac{1}{2} \frac{k'_{<}}{\overline{k}_{<}} \right) = \frac{1}{2} \overline{k}_{<}^{1/2} \left(1 + \frac{k_{<}}{\overline{k}_{<}} \right)$$

Extra terms $C_{pn} \sum_{m=1}^{N} (\lambda_m + |a_m|^2) a_n$

- ν_T bifurcation parameter for model
- Extra information needed to determine ν_T (e.g. DNS)

The effect of unresolved scales

- Coherent part with cubic term and linear term L^{ν}. If L^{ν} has the same structure as L^{D}:

L^{D} nearly diagonal and negative $\Longrightarrow L^{\nu}_{np} = \delta_{np} L^{\nu}_{pp} = -\alpha_p \delta_{np} \sum \lambda_i$, $C_{np} = -\alpha_p \delta_{np}$

- Incoherent part modelled by noise ϵ_p

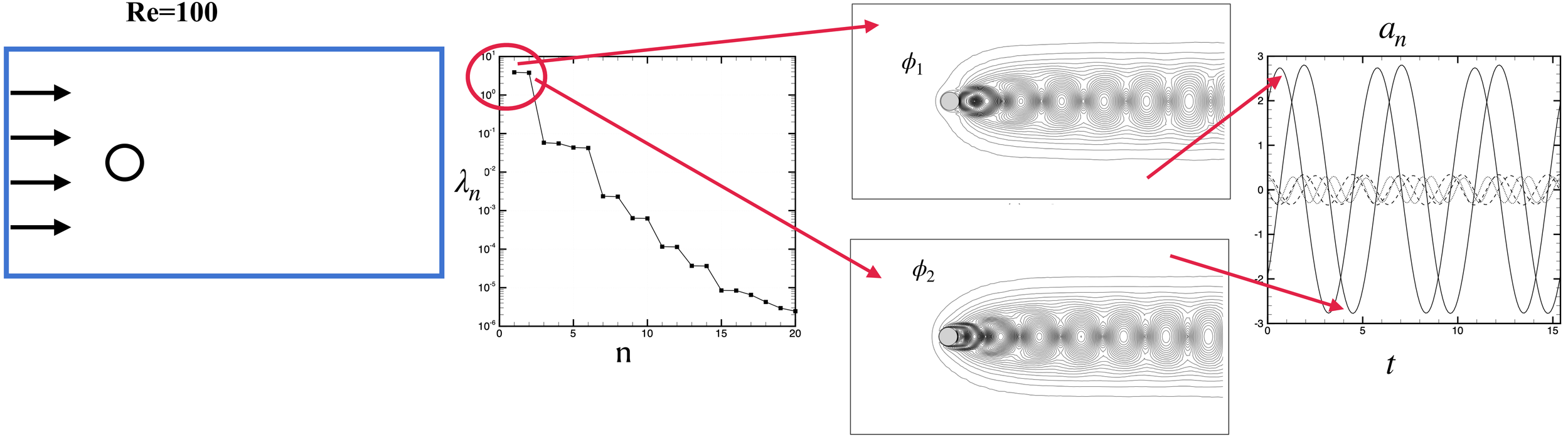
$\Longrightarrow$ Final form of the model:

$$\dot{a}_p = L_{np} a_n + Q_{nmp} a_n a_m - \alpha_p a_i^2 a_p + \epsilon_p \text{ with } L = L^D + L^M + L^{\nu}$$

where α_p is determined from observations of $a_n(t)$

A first example: cylinder flow

2-D flow behind a cylinder (Cordier and Bergmann 06)

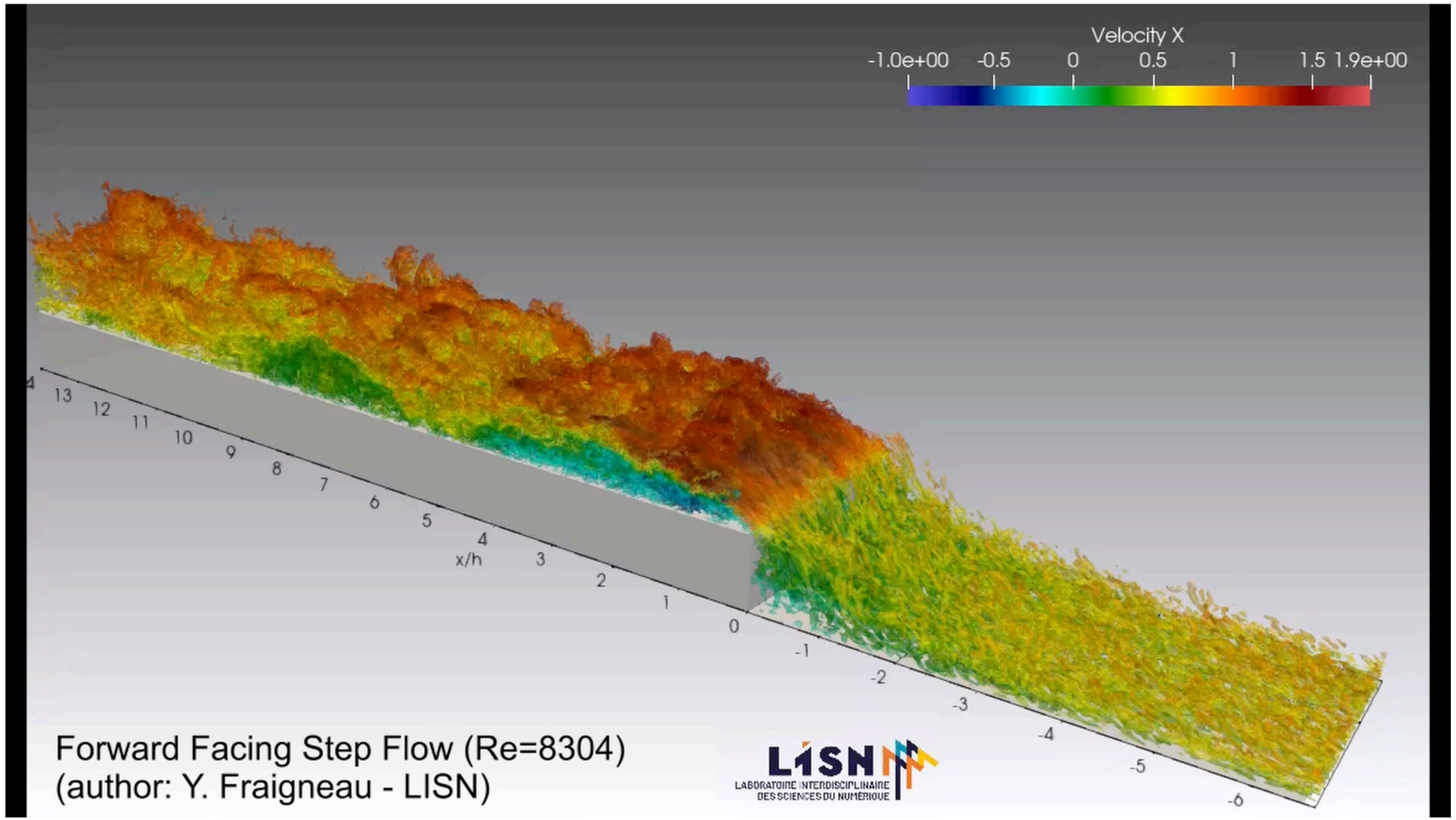


Stabilized two-mode model:

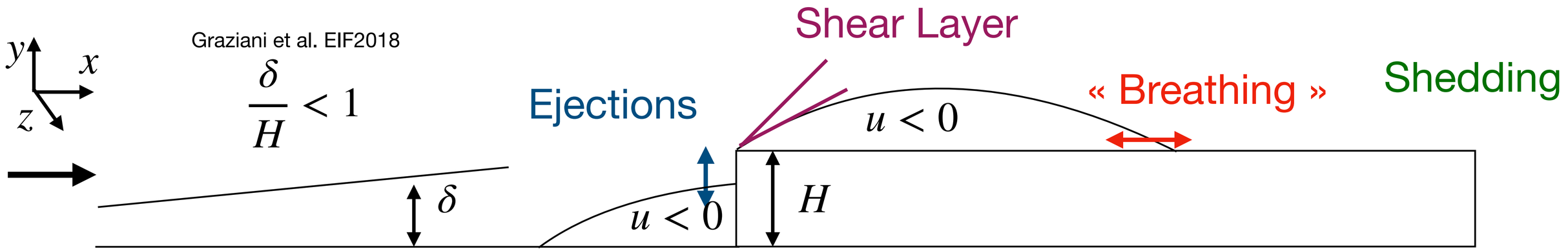
$$\dot{a}_1 = \omega\, a_2 + C(A^2 - |a_1|^2 - |a_2|^2)a_1$$

$$\dot{a}_2 = -\omega\, a_1 + C(A^2 - |a_1|^2 - |a_2|^2)a_2$$

A second example: the forward-facing step



The key dynamics

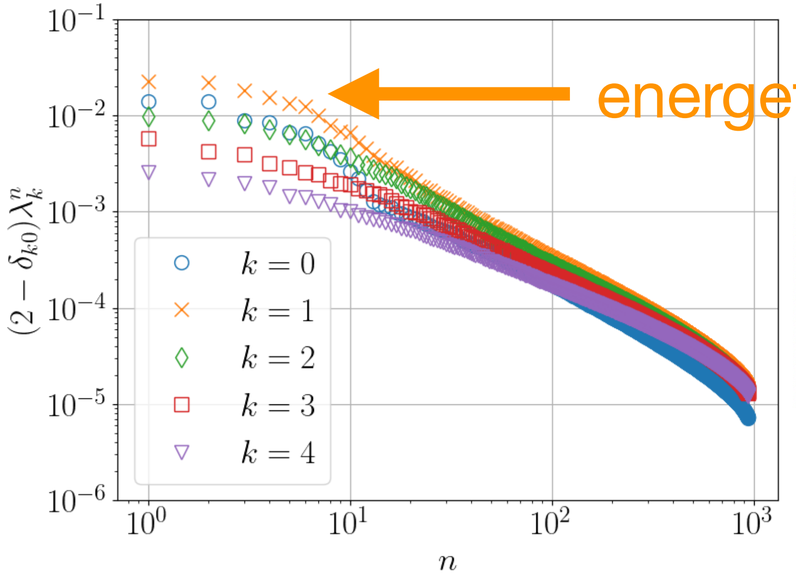


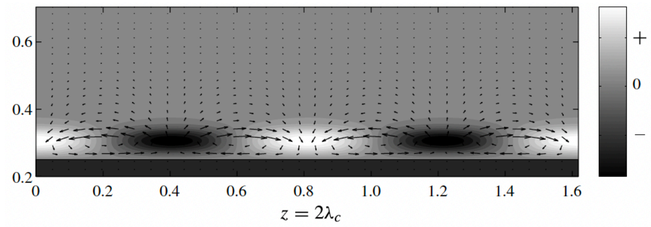
Collab PPRIME (F. Kerhervé), U. Calgary (E. Larose, R. Martinuzzi)

Homogeneity in spanwise direction

$\Longrightarrow$ Fourier representation in z:

$$u(x,y,z,t) = \sum_k \sum_n a_k^n(t)\phi_k^n(x,y)e^{2i\pi z/L_z}$$





Lanzerstorfer and Kuhlmann, JFM 2012

A basic model

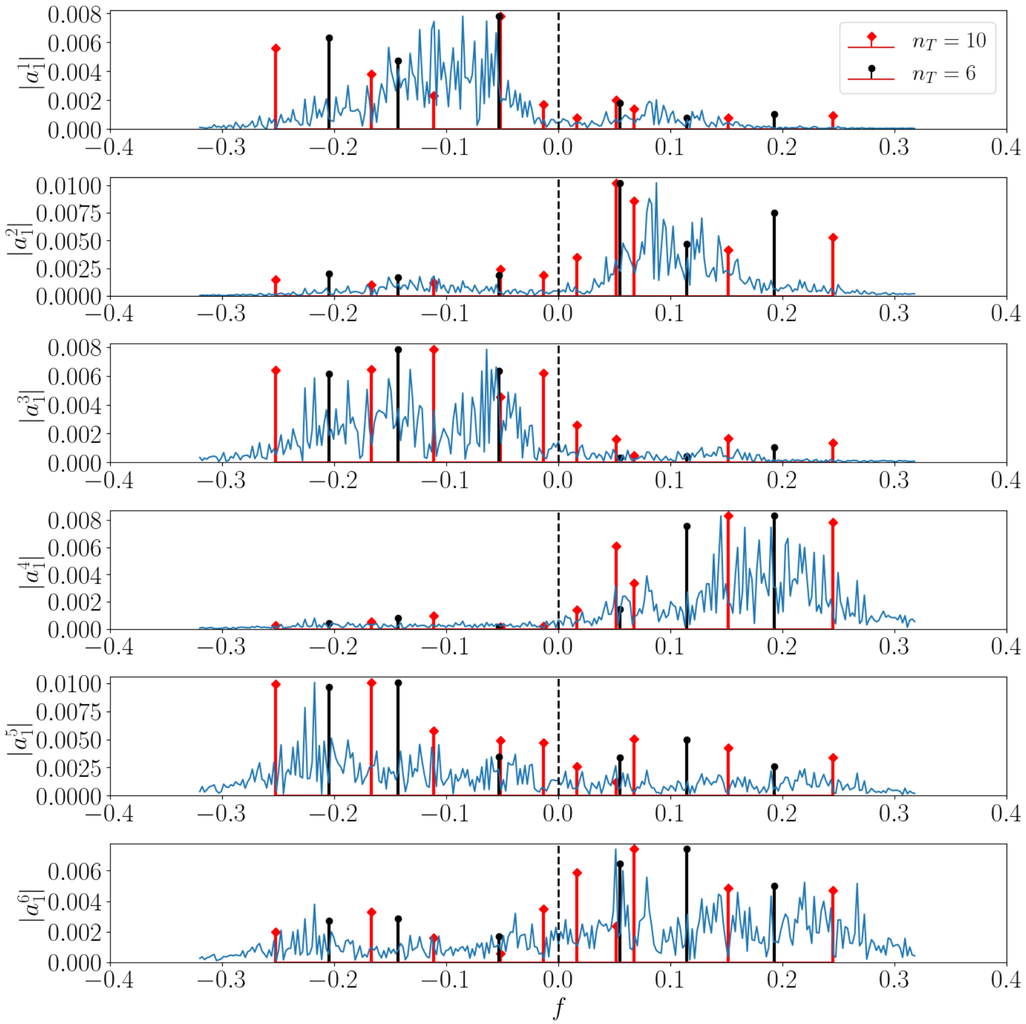
Model with constant mean flow
(does not account for ejections)

$$\rightarrow \quad \dot{a}_k^n = L_{nm} a_k^m$$

Results robust w.r.t $n_T \geq 6$

Discrete frequencies match DNS frequency spectrum

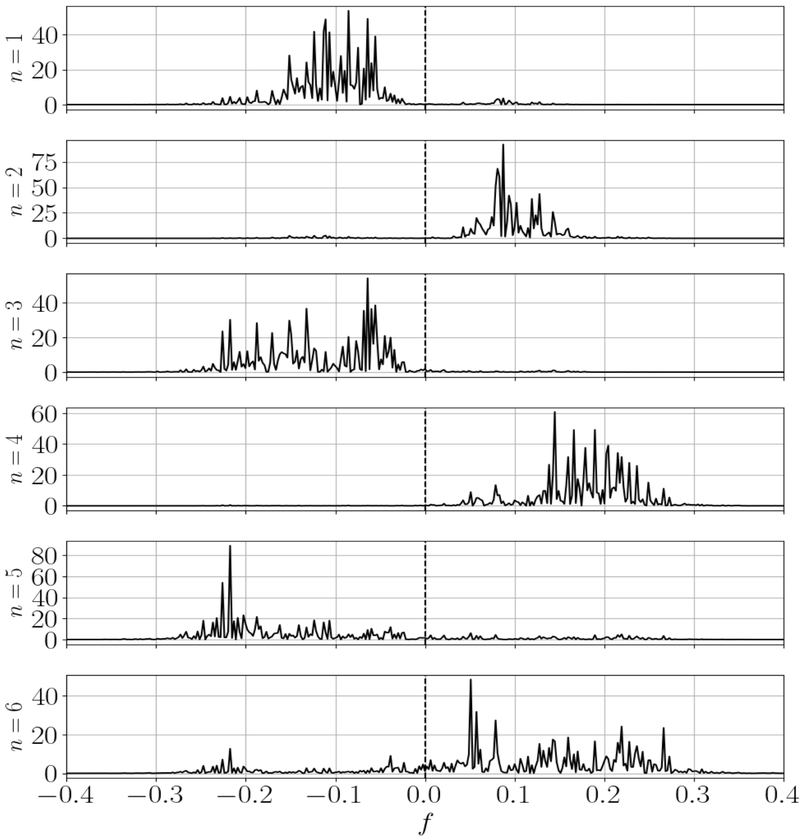
$$|\hat{a}_1^n(f)|^2$$



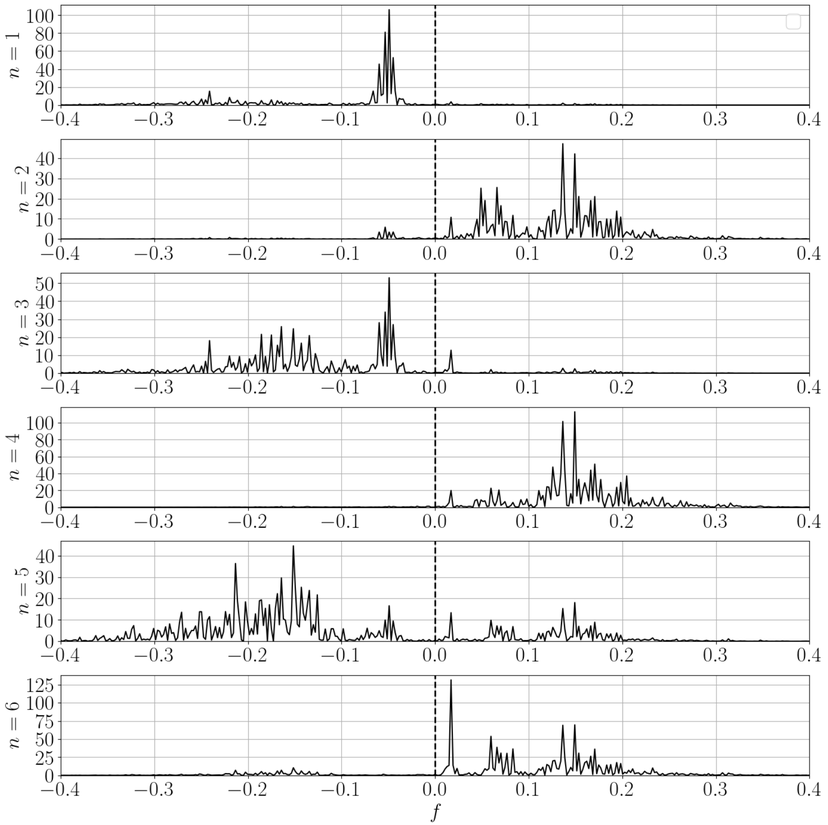
A (slightly less) basic model

$\rightarrow \dot{a}_k^n = \tilde{L}_{nm}(t) a_k^m$ where $\tilde{L}$ varies like the maximum shear at the edge

DNS

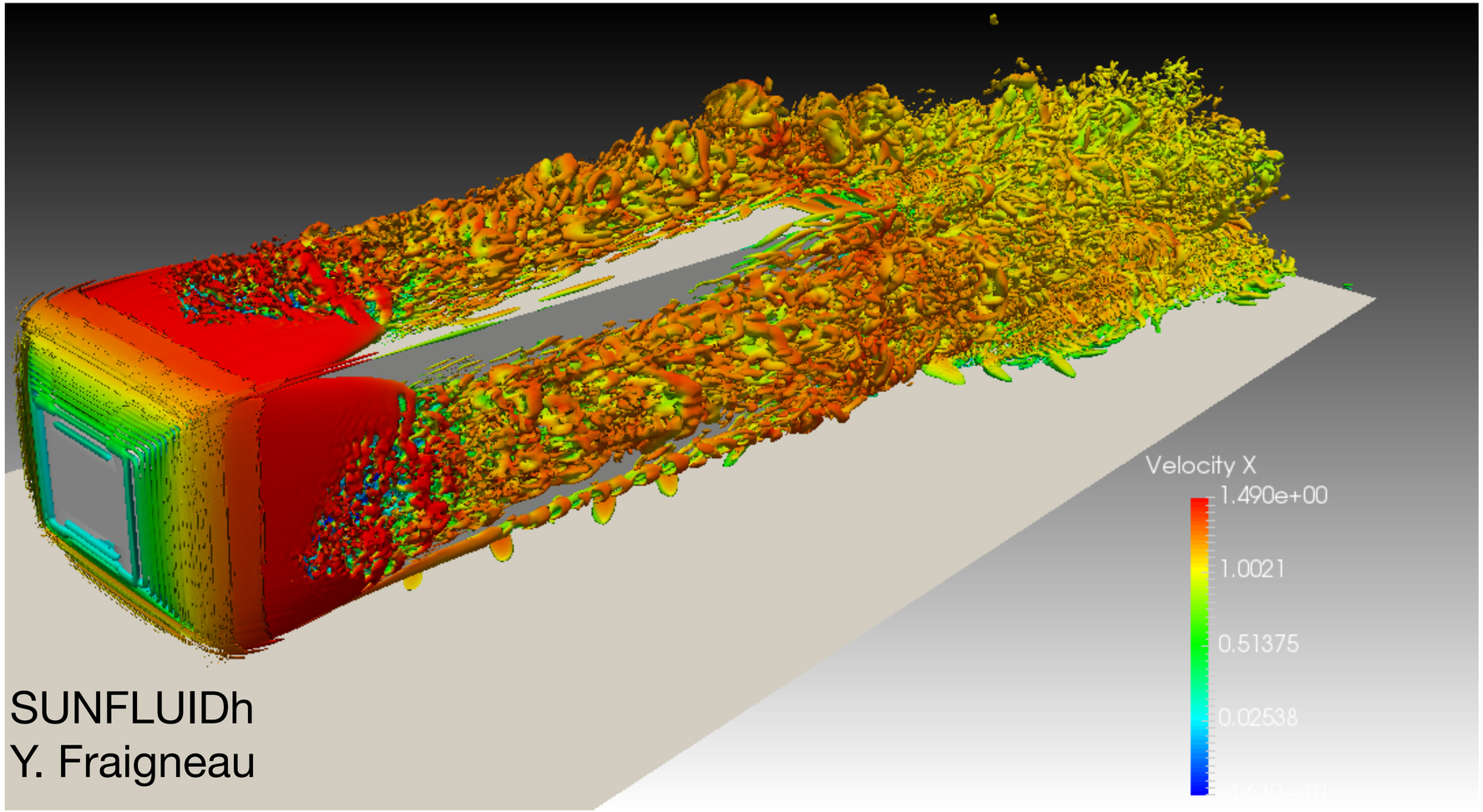


Non-autonomous model



Correct spectral content

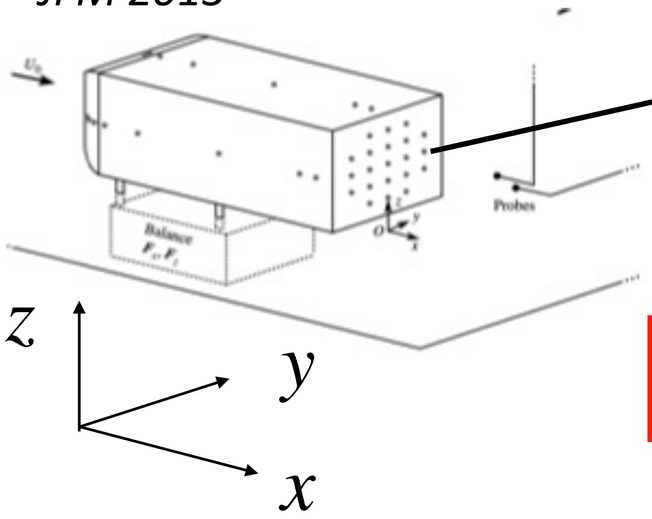
A third example: bluff-body wake



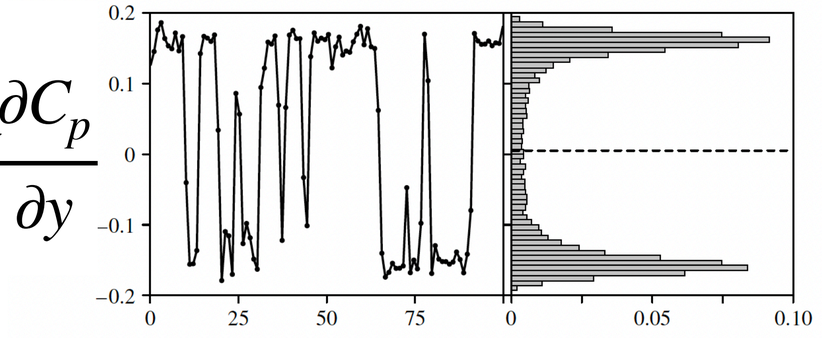
Ahmed body

A wide diversity of time scales

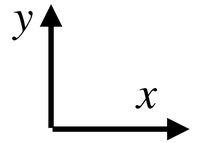
Grandemange and Cadot
JFM 2013



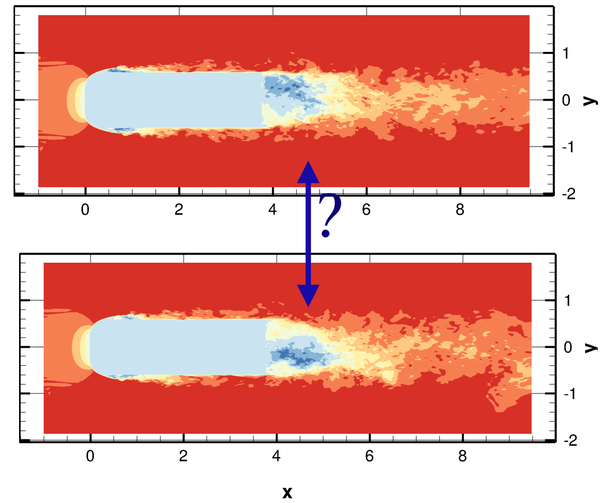
$$\frac{\partial C_p}{\partial y}$$



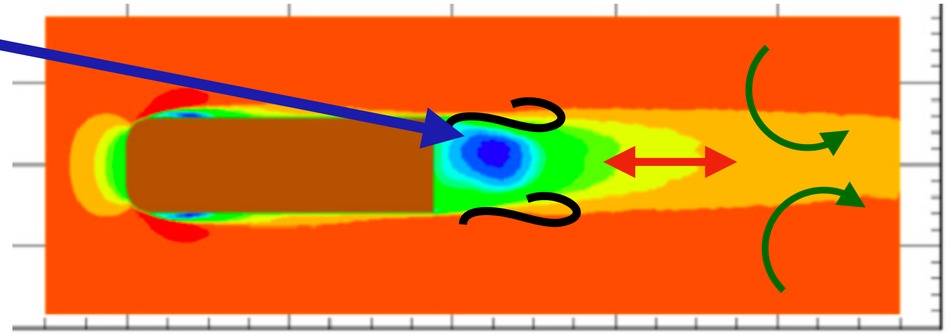
Intermittent switches between quasi-stationary wake symmetry-breaking states



Top view



- Wake deviation switch tU/H~O(1000)
- Wake pumping. tU/H~ O(10-20)
- Vortex shedding tU/H~ O(5)
- Shear layer instability tU/H ~ O(1)



Can we build a model that captures the switches?

POD analysis

$$u(x,t) = \sum_n a_n(t)\phi_n(x)$$

Decreasing energy λ_n

Deviation

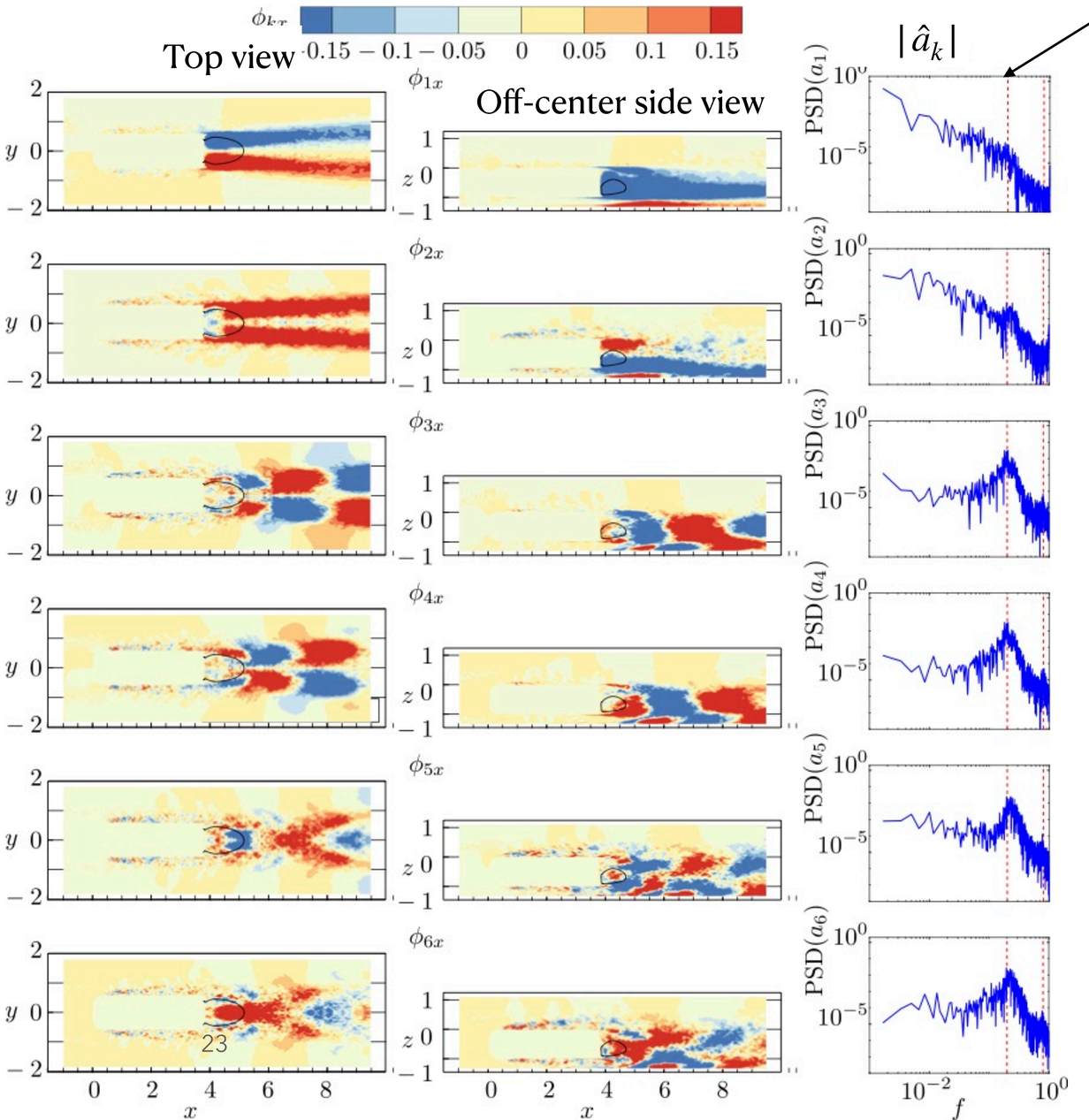
« Switch »

Lateral Shedding

Lateral Shedding

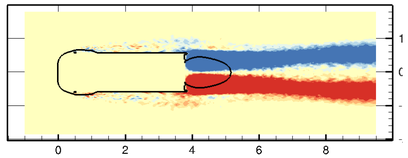
Vertical Shedding

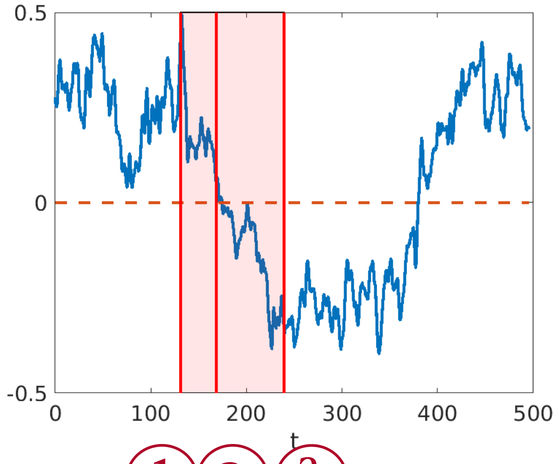
Vertical Shedding



Evolution of the dominant POD modes during a switch

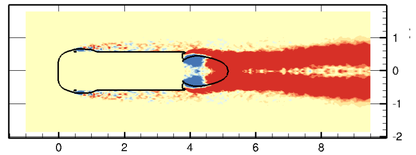
a_1

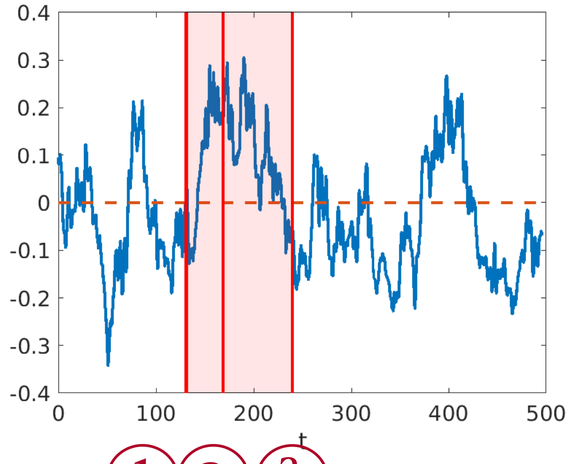




1 2 3

a_2





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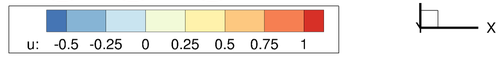
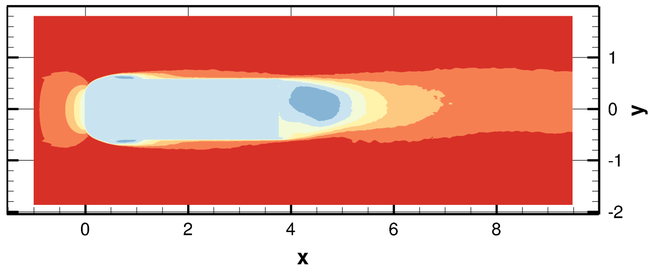
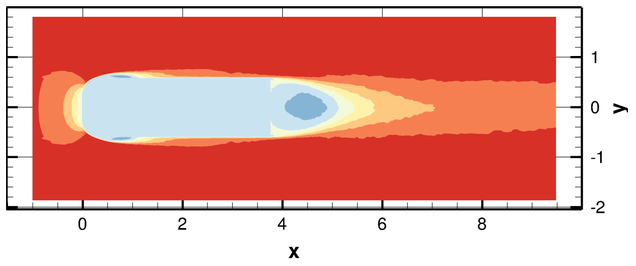


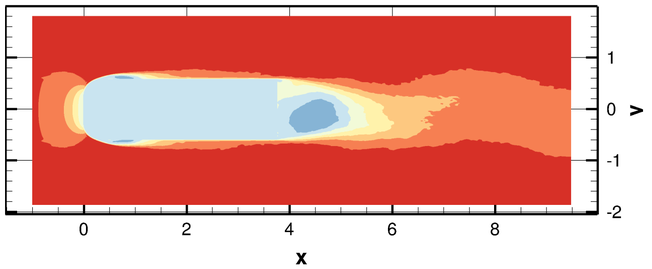
Flow reconstructions based on mean and 6 fluctuating modes



1



2



3

A nonlinear model

$\dot{a}_0 = 0\ (a_0 = 1)$ Mean

$\dot{a}_1 = A_1 + 0.011a_1a_2 + 0.019a_0a_3 + 0.023a_0a_4 + 0.045a_3a_5 + 0.005a_1a_6 + \epsilon_1$

$\dot{a}_2 = A_2 + 0.12a_0a_5 + 0.11a_0a_6 + 0.007a_2a_3 - 0.018a_1^2 + 0.018a_0^2 + \epsilon_2$

$\dot{a}_3 = A_3 + 1.08a_0a_4 - 0.02a_0a_1 + 0.037a_1a_6 - 0.025a_2a_4 - 0.02a_0a_3 + \epsilon_3$

$\dot{a}_4 = A_4 - 1.08a_0a_3 - 0.16a_0a_1 - 0.04a_1a_5 + 0.025a_2a_3 - 0.02a_0a_4 + \epsilon_4$

$\dot{a}_5 = A_5 + 1.18a_0a_6 - 0.13a_0a_2 + 0.035a_2a_3 - 0.013a_1a_3 + 0.029a_1a_4 - 0.013a_2^2 - 0.02a_0a_5 + \epsilon_5$

$\dot{a}_6 = A_6 - 1.18a_0a_5 - 0.26a_0a_2 - 0.026a_2a_5 - 0.02a_1a_3 - 0.02a_0a_6 + \epsilon_6,$

Von Karman shedding frequencies

Reflectional symmetry determines form of quadratic terms

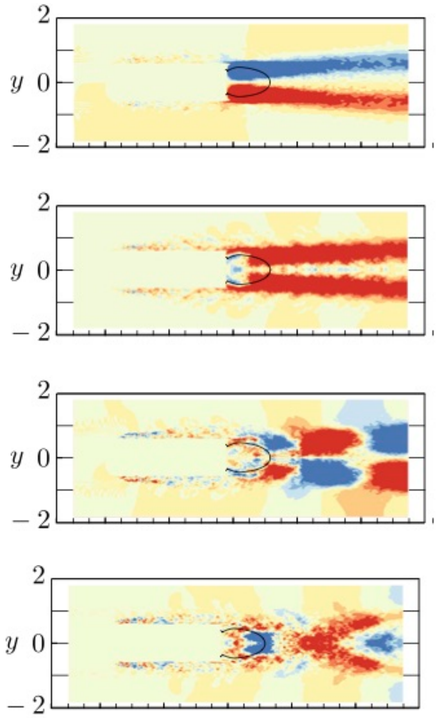
Long-time integration of the model

1: Deviation

2: Switch

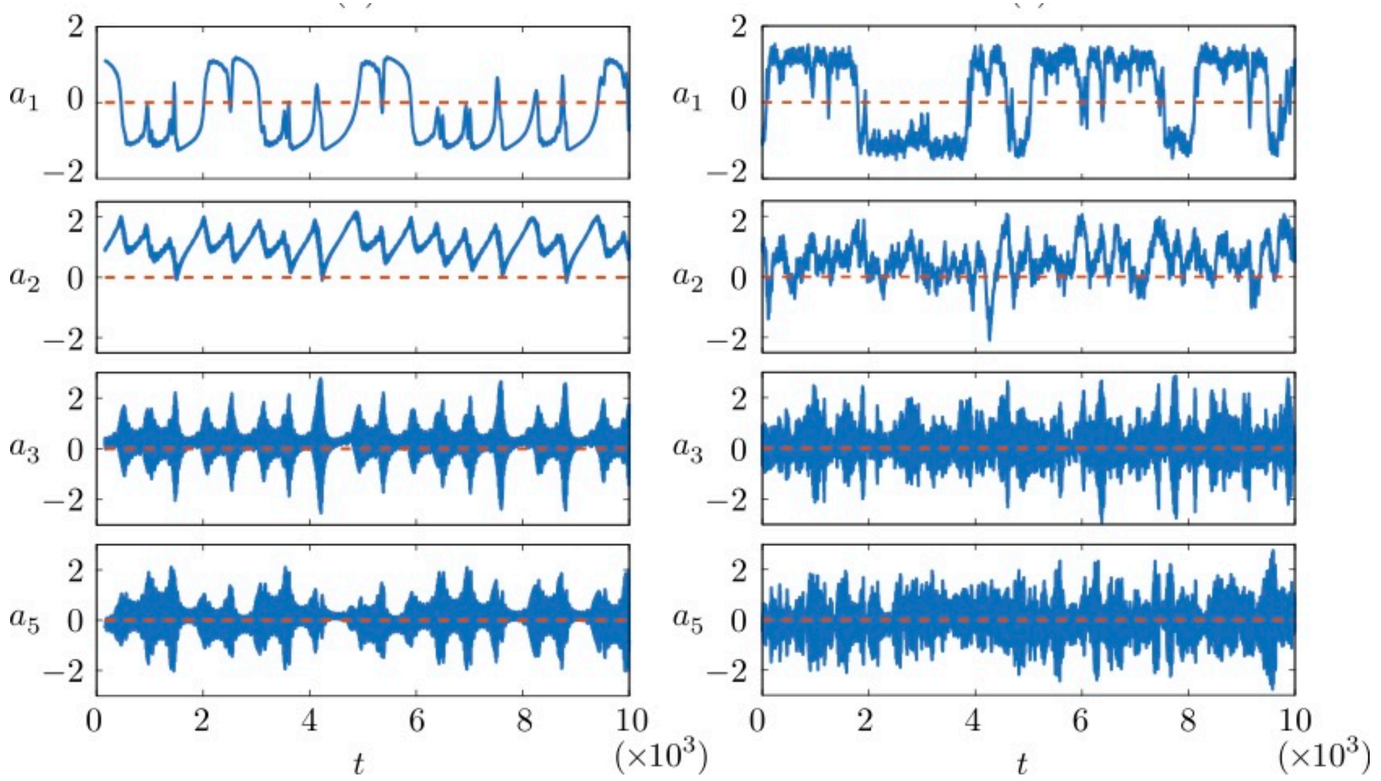
3: Lateral VK

5: Vertical VK



Without noise $\sigma = 0$

With noise $\sigma = 0.07$



Model produces switches that are triggered by excitation of shedding modes.

Addition of noise improves agreement with DNS.

Summary

- POD-based models can capture some key dynamics of complex, high-dimensional separated flows (also convection problems!)
- More work is needed to improve the models
- Machine learning opens new possibilities to better link data and physics
    - improve small-scale modelling
    - handle missing information

References

Low-dimensional analysis and modelling of the flow over a forward-facing step. B. Podvin, **Y. Fraigneau**, Journal of Fluid Mechanics 2024

Low-order modelling of the wake dynamics of an Ahmed body, B. Podvin, **S. Pellerin, Y. Fraigneau, G. Bonnavion, O. Cadot**, Journal of Fluid Mechanics 2021

Low-order models for predicting radiative transfer effects on Rayleigh-Bénard convection in a cubic cell at different Rayleigh numbers, **L. Soucasse**, B. Podvin, **P. Rivière, A. Soufiani,** Journal of Fluid Mechanics 2021

Thank you