Apport des modèles réduits paramétriques pour les problèmes inverses en thermique

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Journéee SFT, Mesures Thermiques et Techniques Inverses: Approches originales en inversion de mesures - méthodes et applications
11 Juin 2026

Context

Physical phenomena

1D heat diffusion equation:

$$c(T, x, t)\frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left(\lambda(T, x, t)\frac{\partial T}{\partial x}\right), \quad x \in \left[0, L\right], t > 0, \tag{1}$$

with

- $c\left[\mathrm{J.m^{-3}.K^{-1}}\right]$ the volumetric heat capacity
- $\lambda\left[\mathrm{W.m^{-1}.K^{-1}}\right]$ the thermal conductivity

Robin-type boundary conditions:

$$\lambda(T, x, t)\frac{\partial T}{\partial x} = h_L(T, t)\left(T - T_L(t)\right) - q_L(T, t), \quad x = 0, t \geqslant 0,$$

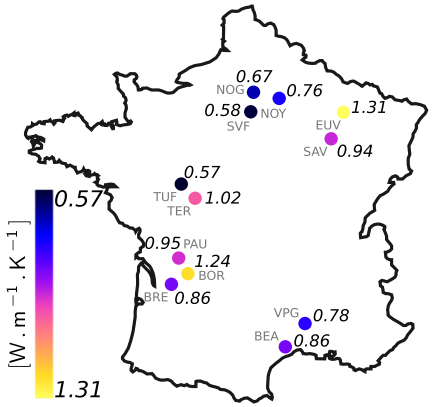
$$-\lambda(T, x, t)\frac{\partial T}{\partial x} = h_R(T, t)\left(T - T_R(t)\right) - q_R(T, t), \quad x = L, t \geqslant 0$$

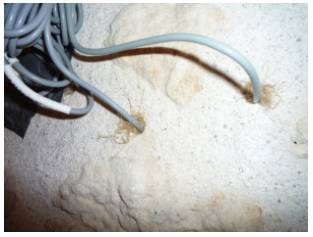
Initial condition:

$$T(x, t) = T_0(x), \quad x \in \left[0, L\right], t = 0,$$

Context

Inverse Problem: numerical challenges





$$\mathcal{J}(\boldsymbol{p}) = \left|\left| T(x, t, \boldsymbol{p}) - \widehat{T}(t) \right|\right|_2$$

$\rightsquigarrow$ high number of direct problem computations $T(x, t, \boldsymbol{p})$

$\rightsquigarrow$ require sensitivity function $\nabla_{\boldsymbol{p}} T$

⚠ CPU time. 2 options:

- Lumped physical Model
- Reduced Order Model

Context

Reduced Order model (ROM)

- degree of freedom Dof $= N \ll N_x \cdot N_t$
- the fidelity of the physical model is not degraded
- reduced computational cost
- no loss of accuracy

Two approaches:

- a posteriori: requires a *known* solution (or experimental data)
- a priori: no requirement

Basic idea: Decomposition of the solution

$$T(x, t, \boldsymbol{p}) \approx \sum_{n=1}^{N} \varphi_n(x)\, a_n(t)$$

Methods: MIM, POD, PGD, Spectral ROM, etc.

State-of-the-art: inverse problem + ROM

- [1] MIM + Frequentist
- [2] MIM (T & sensibility) + Frequentist
- [3, 4] PGD + Bayesian
- ...

ROM parametric solutions:

Solution depends explicitly on extra-parameters of the problem $T(x, t, \boldsymbol{p})$

Inverse Problem + ROM

Problematic

Inverse problem $\mathcal{J}(\boldsymbol{p}) = \left|\left| T(x, t, \boldsymbol{p}) - \widehat{T}(t) \right|\right|_2$

$\rightsquigarrow$ high number of direct problem computations $T(x, t, \boldsymbol{p})$

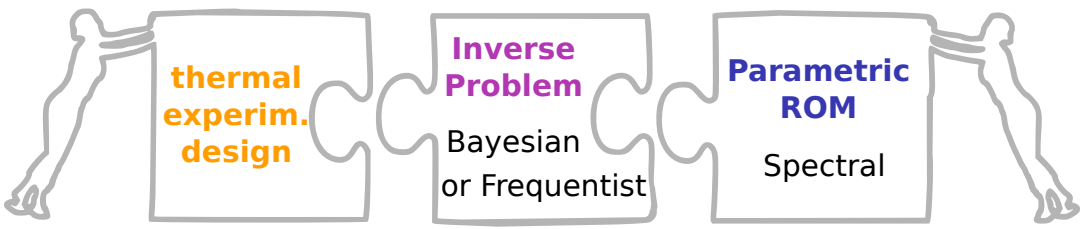
$\rightsquigarrow$ require sensitivity function $\boldsymbol{\nabla}_{\boldsymbol{p}} T$

$+$

ROM parametric solutions $T(x, t, \boldsymbol{p})$

$=$

How to use parametric solutions in the framework of inverse problems?



Frequentist estimation

methodology

GAUSS algorithm framework [? ?]:

$$p^{(k+1)} = p^{(k)} + (\boldsymbol{X}^\top \boldsymbol{X})^{-1} \left(\boldsymbol{X}^\top \left(\widehat{\boldsymbol{T}} - \boldsymbol{T}(p^{(k)}) \right) \right),$$

Spectral parametric solution [5]

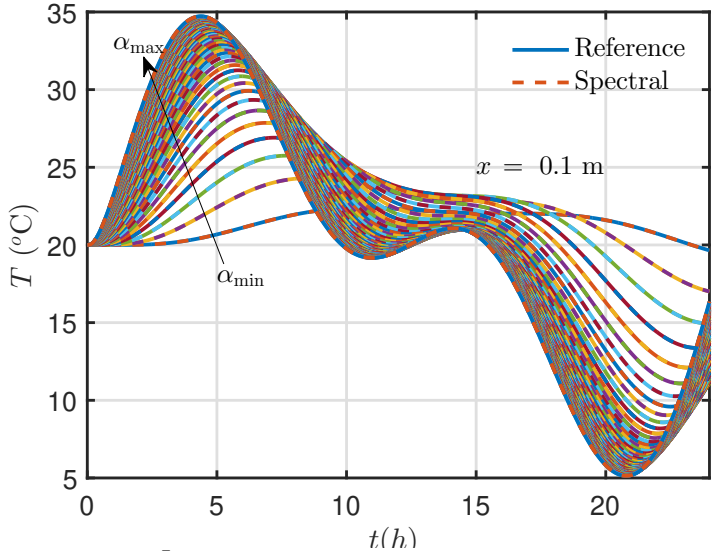
$$T(x, t, p) = \sum_{n=1}^{N} \sum_{m=1}^{M} \mathsf{T}_n(x)\, a_{nm}(t)\, \mathsf{P}_m(p)$$

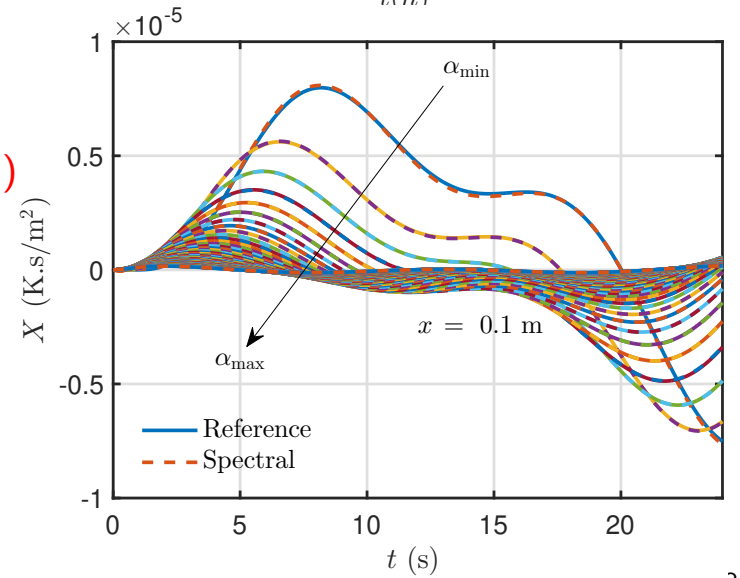
and the sensitivity function:

$$X(x, t, p) = \frac{\partial T}{\partial \bar{p}} = \sum_{n=1}^{N} \sum_{m=1}^{M} \mathsf{T}_n(x)\, a_{nm}(t)\, \frac{\mathrm{d}\mathsf{P}_m}{\mathrm{d}p}(p)$$

⇝ Fast computations:

- of T
- and of $\boldsymbol{X} = \boldsymbol{\nabla}_p T$





Frequentist estimation

methodology

Modified GAUSS algorithm using Spectral parametric solution:

off-line phase

1. build the Spectral parametric solution for Eq. (1)
2. save the spectral coefficients $\left\{ a_{ij} \right\}$

on-line phase

1. Set $p^{(0)}$
2. While $\mathcal{J}^{(k)} \geqslant$ Tol
   a. Evaluate $T(p^{(k)})$ and $X(p^{(k)}) \equiv$ compose the solution
   b. Compute $p^{(k+1)}$
   c. Compute $\mathcal{J}(\, p^{(k+1)}\,)$
   d. increment $k \leftarrow k + 1$

$$T(x, t, p) = \sum\sum \mathrm{T}_n(x)\, a_{n\,m}(t)\, \mathrm{P}_m(p)$$

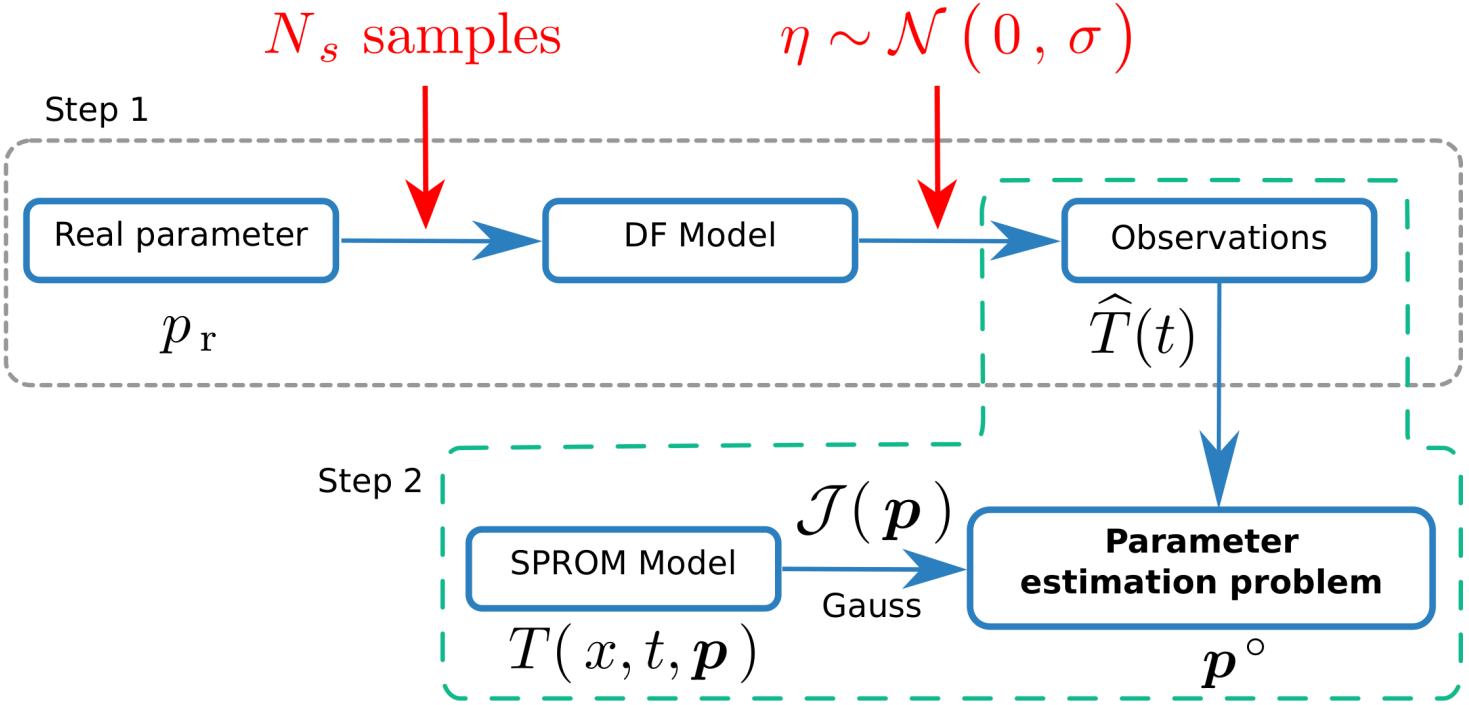
$$X(x, t, p) = \sum\sum \mathrm{T}_n(x)\, a_{n\,m}(t)\, \frac{\mathrm{dP}_m}{\mathrm{d}p}(p)$$

Frequentist estimation

Case study

inverse crime

- $N_s = 10^4$ samples
- $0.5\,^\circ C$ noise
- $\alpha^{\mathrm{r}} = 0.55\ \mathrm{mm}^2\,.\,\mathrm{s}^{-1}$



Frequentist estimation

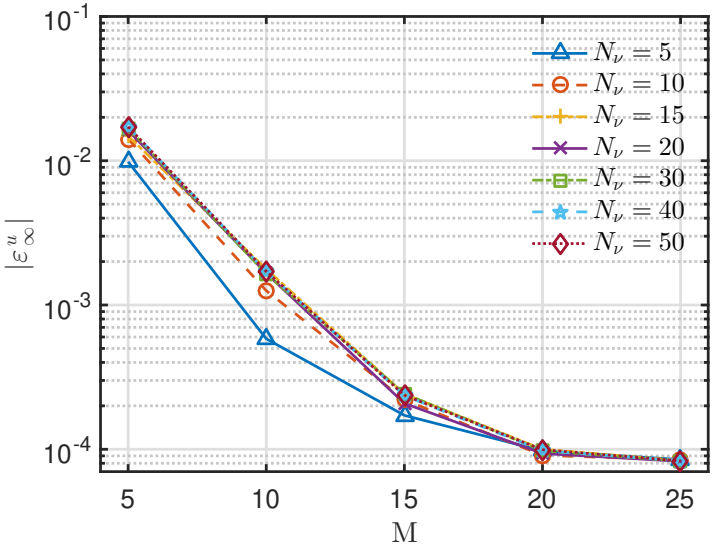
off-line computation

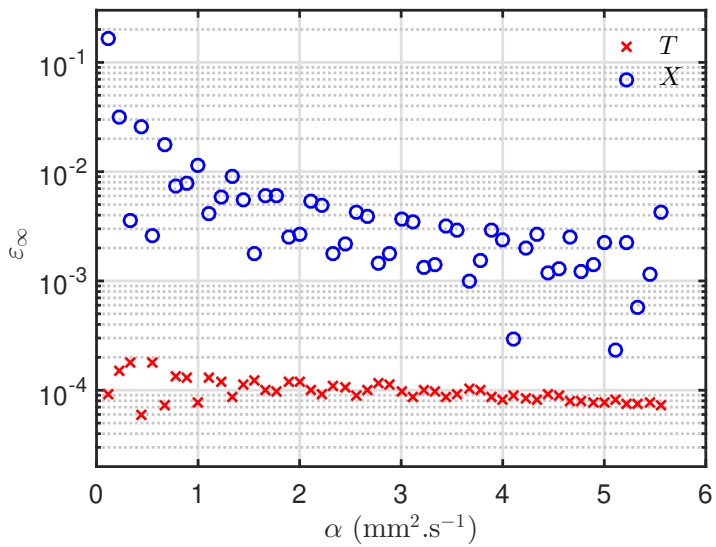
Spectral parametric solution

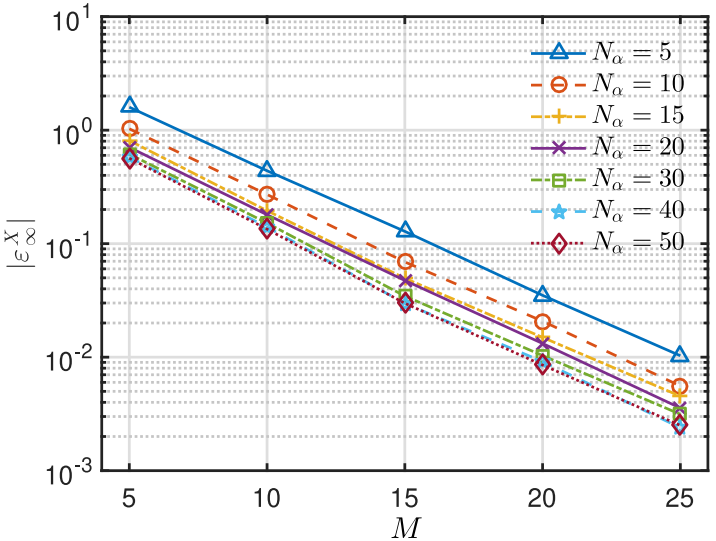
$$T(x,t,p) = \sum_{n=1}^{N} \sum_{m=1}^{M} \mathsf{T}_n(x)\, a_{nm}(t)\, \mathsf{T}_m(\alpha)$$

$$X(x,t,p) = \frac{\partial T}{\partial \bar{p}} = \sum_{n=1}^{N} \sum_{m=1}^{M} \mathsf{T}_n(x)\, a_{nm}(t)\, \frac{\mathrm{d}\mathsf{T}_m}{\mathrm{d}p}(\alpha)$$

- $N \geqslant 18$ for convergence (initial condition)
- $M = 20$ at least for sensitivity
- $\alpha \in [0.1, 5.5]\ \mathrm{mm}^2.\mathrm{s}^{-1}$

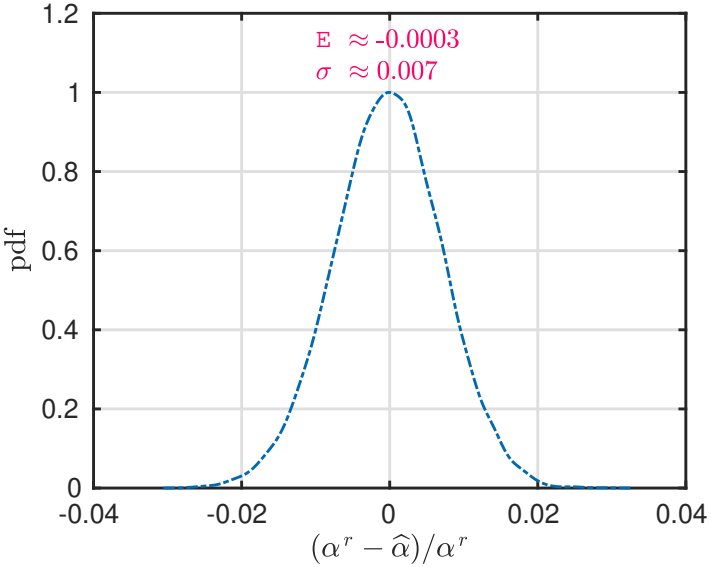


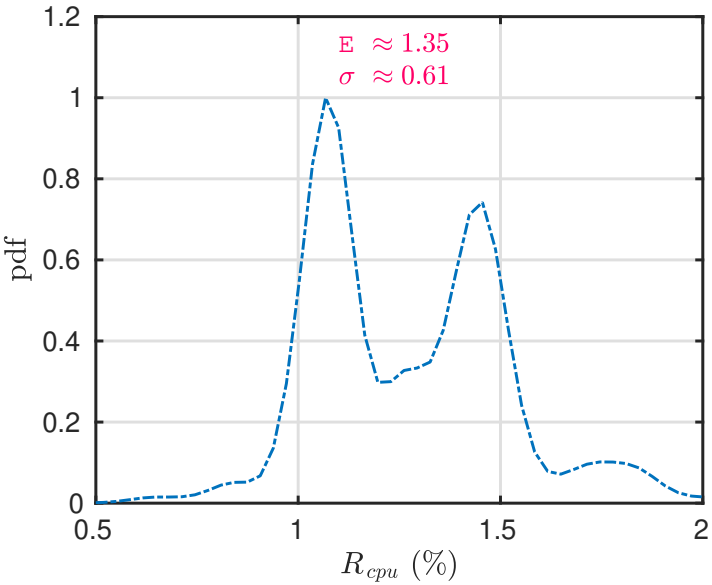


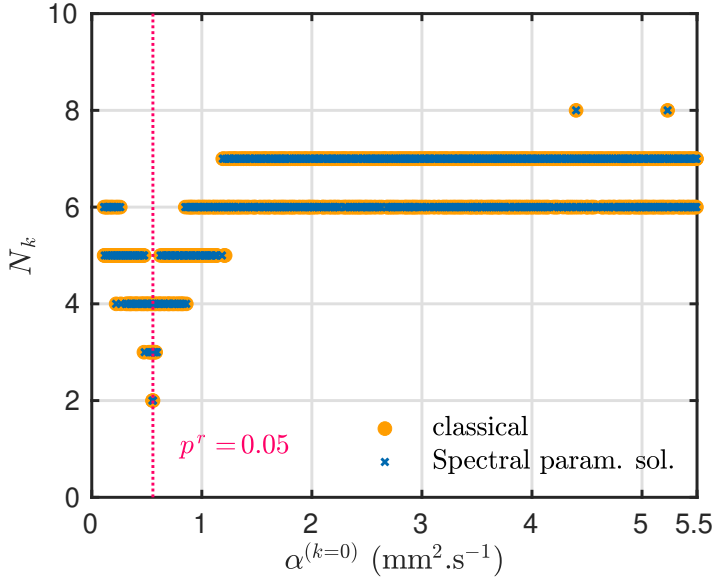


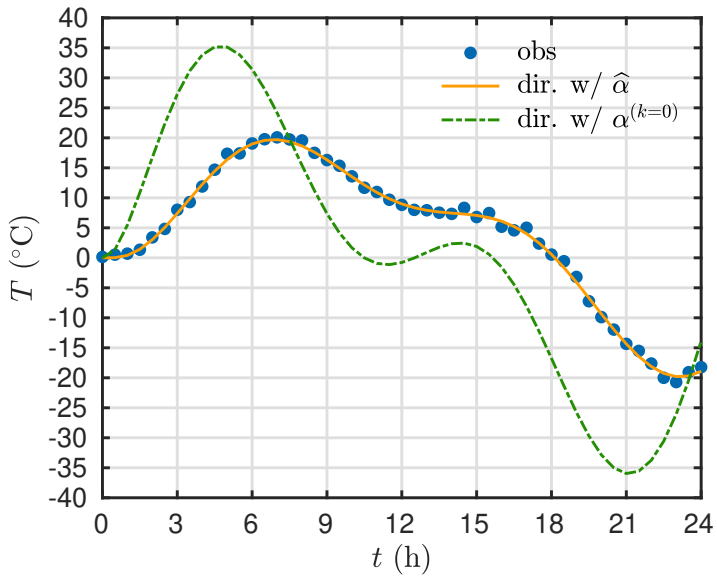
Frequentist estimation

results









Bayesian estimation [6]

framework

Baye's theorem:

$$\pi\left(\boldsymbol{p}\,\middle|\,\widehat{T}\right) = \frac{\pi\left(\boldsymbol{p}\right)\cdot\pi\left(\widehat{T}\,\middle|\,\boldsymbol{p}\right)}{\pi\left(\widehat{T}\right)}$$

- $\pi\left(\boldsymbol{p}\right)$ is the prior probability density of the parameters $\boldsymbol{p}$,
- $\pi\left(\widehat{T}\,\middle|\,\boldsymbol{p}\right)$ is the likelihood function,
- $\pi\left(\widehat{T}\right)$ is the marginal probability density of measurements

Assuming that measurement errors are Gaussian random variables, we have:

$$\pi\left(\widehat{T}\,\middle|\,\boldsymbol{p}\right) \propto \exp\left(-\frac{1}{\widehat{\sigma}^2}\left|\left|\, T(x,\,t,\,\boldsymbol{p}) - \widehat{T}(t)\,\right|\right|_2\right)$$

Prior probability density is assumed as a Gaussian distribution:

$$\pi\left(\boldsymbol{p}\right) = \mathcal{N}\left(\boldsymbol{\mu},\,\boldsymbol{\sigma}\right)$$

Bayesian estimation

methodology

Then, the posterior distribution is given by:

$$-\ln\left(\pi\left(\boldsymbol{p}\,\middle|\,\widehat{T}\right)\right) \propto \frac{1}{\widehat{\sigma}^2}\left|\left|\, T(x,\,t,\,\boldsymbol{p}) - \widehat{T}(t)\,\right|\right|_2 + \sum_{\cdot}\left(\frac{p_k - \mu_k}{\sigma_{p_k}}\right)^2$$

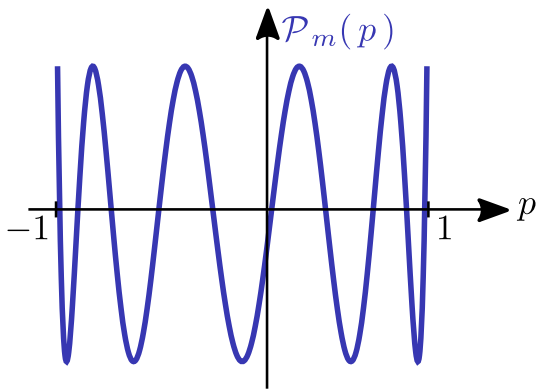
$\Leftrightarrow$ objective function = likelihood + prior distributions

$\rightsquigarrow$ explored with Markov Chain Monte Carlo methods

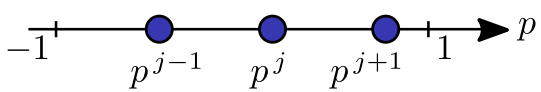
⚠ CPU time, states $\sim 10^6$, 10^7, 10^8

$\rightsquigarrow$ Spectral parametric solution [5]

$$T(x,\,t,\,p) = \sum_{n=1}^{N}\sum_{m=1}^{M} \mathrm{T}_n(x)\, a_{n\,m}(t)\, \mathrm{P}_m(p)$$



global *vs* local



Bayesian estimation

methodology

Modified MCMC algorithm using Spectral parametric solution:

off-line phase

1. build the Spectral parametric solution for Eq. (1)
2. save the spectral coefficients $\left\{ a_{ij} \right\}$

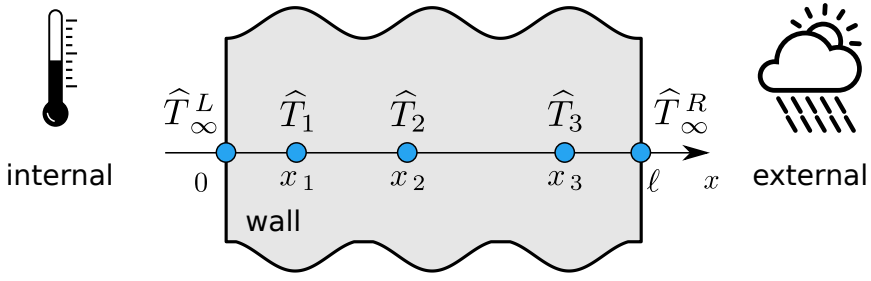
on-line phase

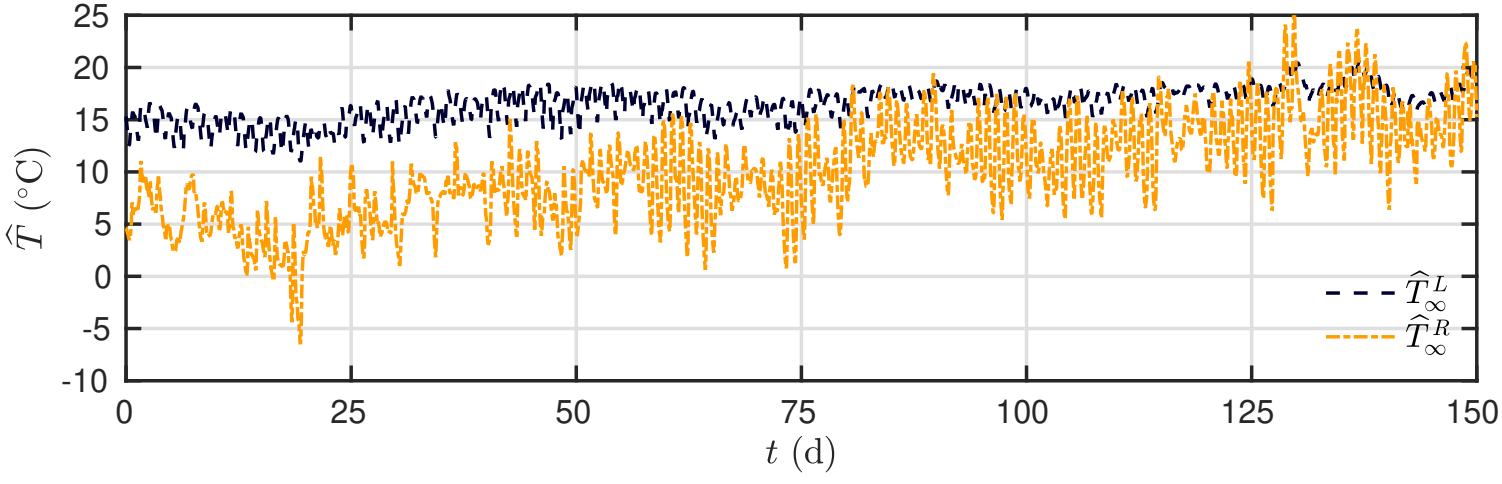
1. While $k \leqslant N_s$
   a. Sample parameter from proposal distribution
   b. Evaluate $T(p^{(k)}) \equiv$ compose the solution
   c. Compute acceptance factor and test with random value

Bayesian estimation

Case study







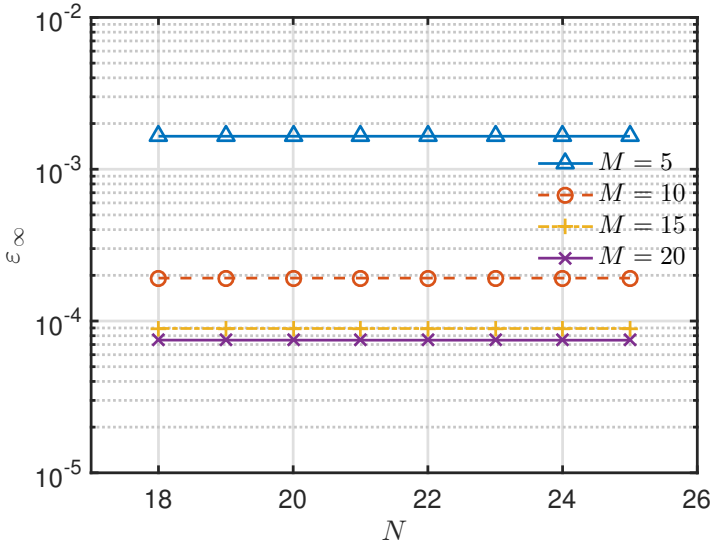
Bayesian estimation

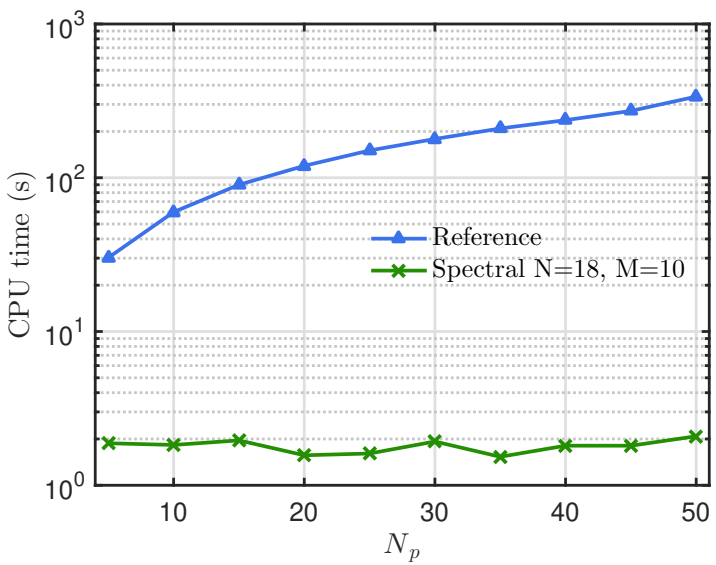
off-line computation

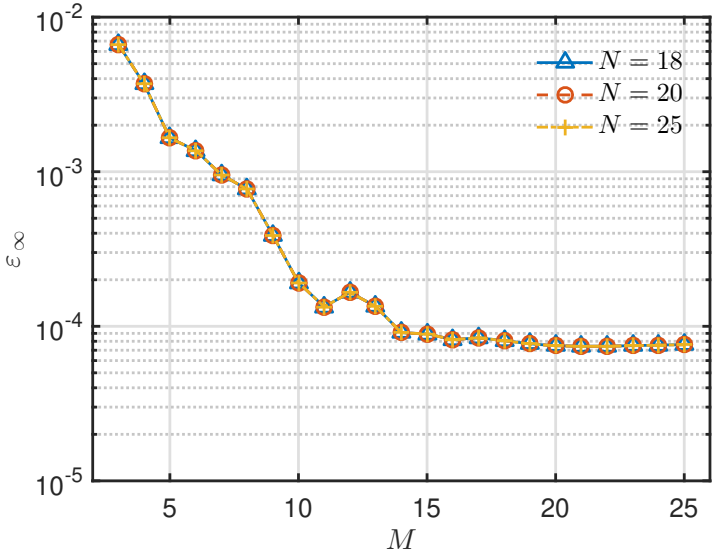
Spectral parametric solution

$$T(x, t, p) = \sum_{n=1}^{N} \sum_{m=1}^{M} \mathsf{T}_n(x)\, a_{nm}(t)\, \mathsf{T}_m(\alpha)$$

- $N \geqslant 18$ for convergence (initial condition)
- $M = 10$ sufficient
- 1 order of CPU time compared to reference

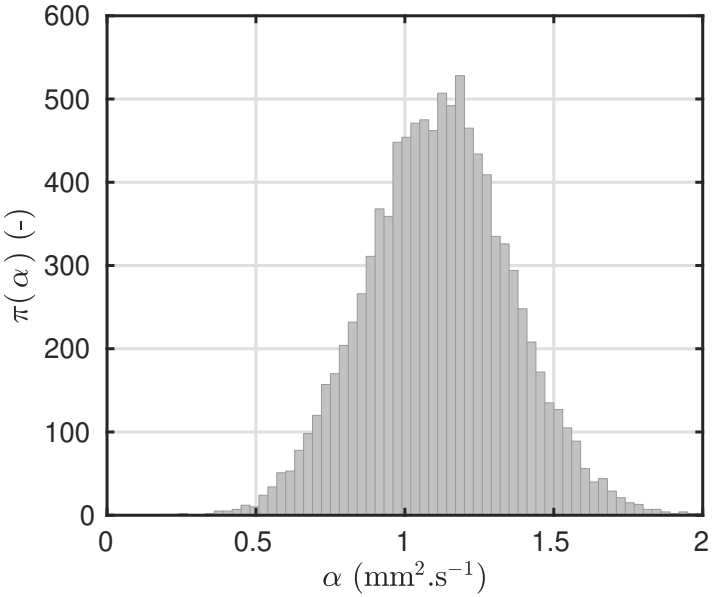


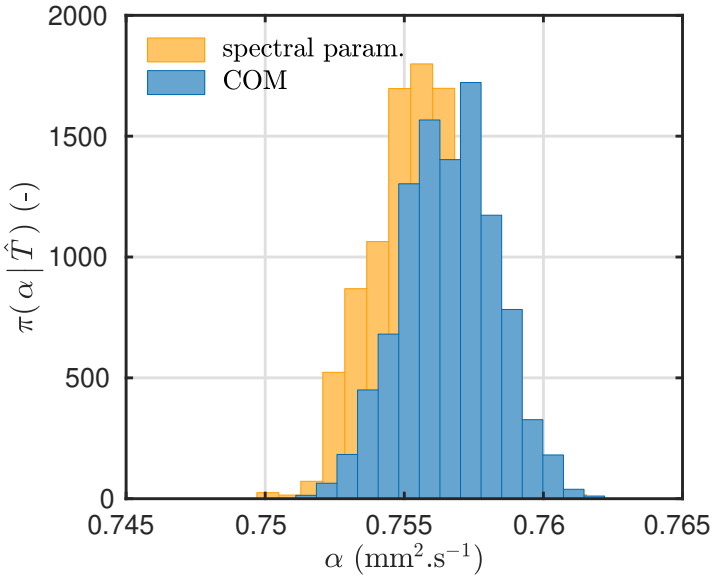


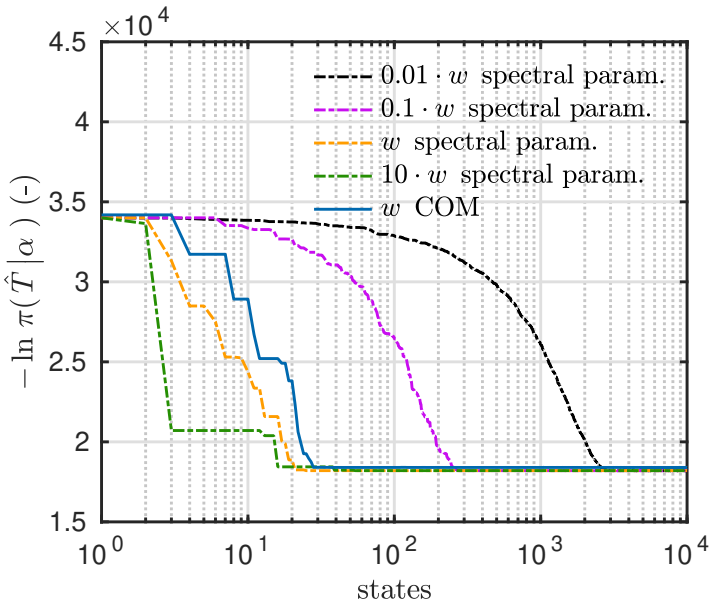


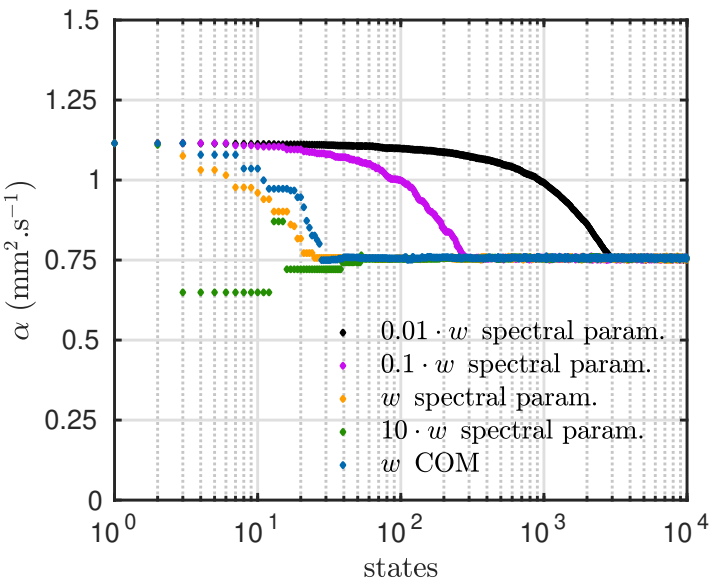
Bayesian estimation

results



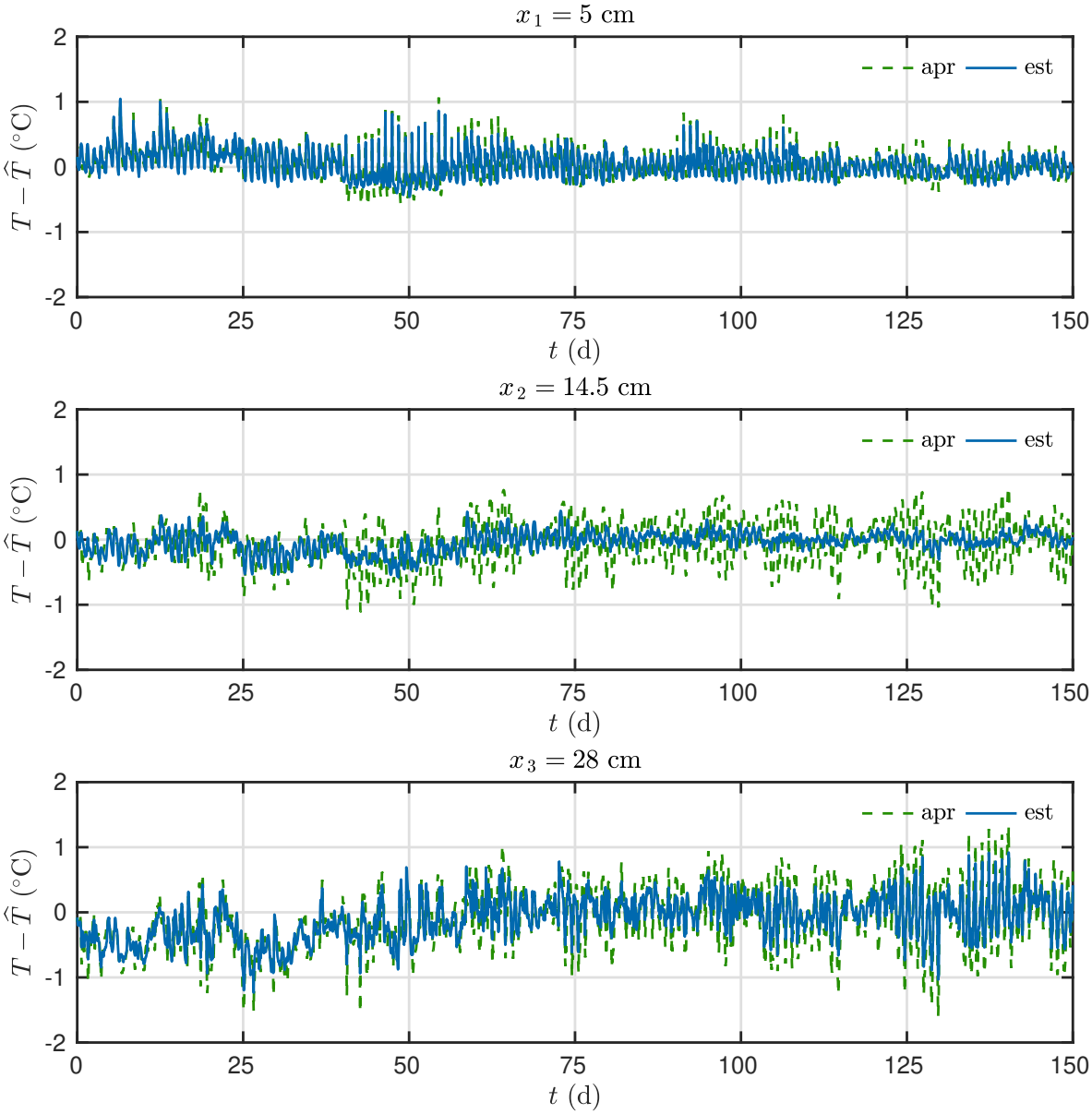






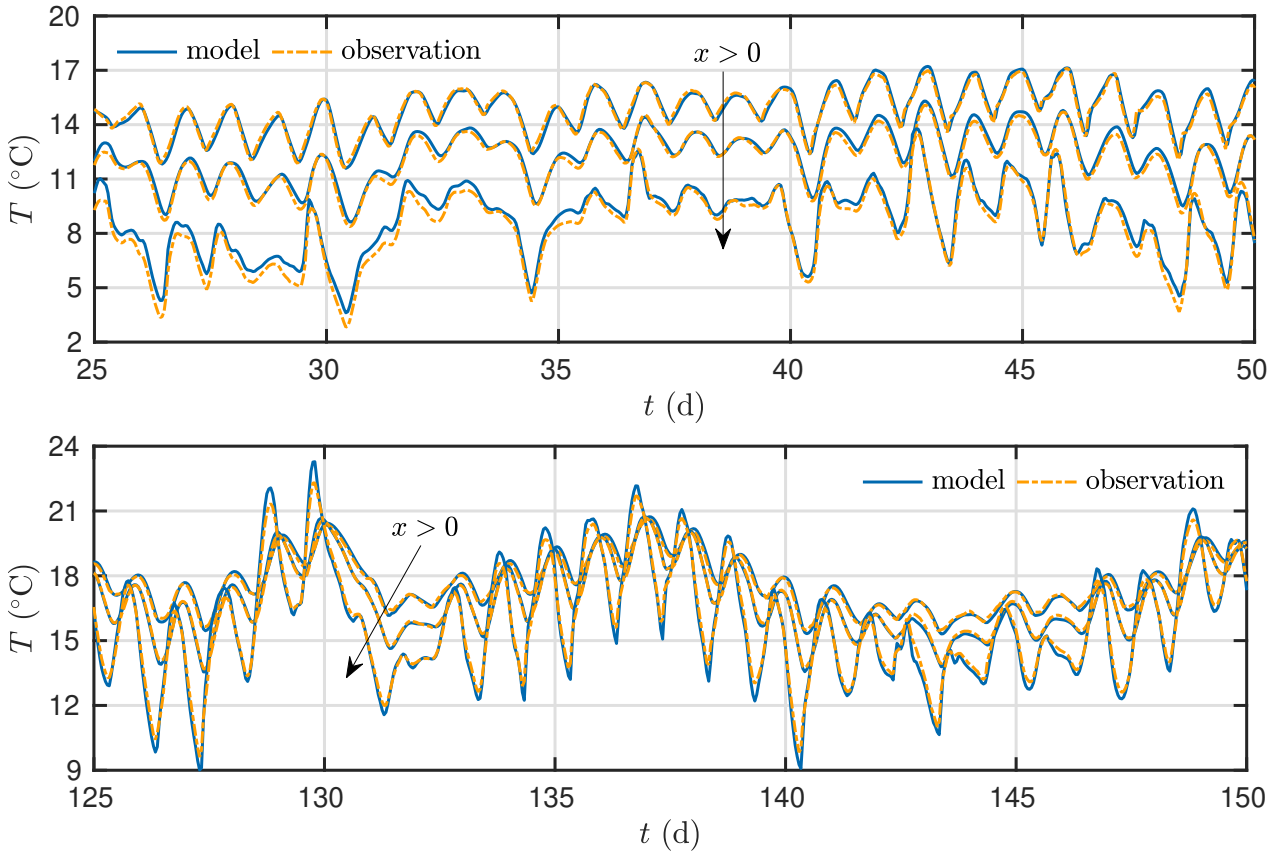
Bayesian estimation

results



Bayesian estimation

residual



Bayesian estimation

CPU time

Thermal Diffusivity $\left[\mathrm{mm}^2 . \mathrm{s}^{-1}\right]$			
	Prior	*Posterior distribution*	
		Spectral parametric model	*Complete Original Model*
Mean	1.11	0.755	0.756
Standard deviation	0.243	$1.71 \cdot 10^{-3}$	$1.68 \cdot 10^{-3}$
Computational time			
		Spectral parametric model	*Complete Original Model*
Off-line		4 min	–
Estimation		8 min	4.6 h

Synthesis

Inverse problem $\mathcal{J}(\boldsymbol{p}) = \left|\left| T(x, t, \boldsymbol{p}) - \widehat{T}(t) \right|\right|_2$

$\rightsquigarrow$ high number of direct problem computations $T(x, t, \boldsymbol{p})$

$\rightsquigarrow$ require sensitivity function $\boldsymbol{\nabla}_{\boldsymbol{p}} T$

$+$

ROM parametric solutions $T(x, t, \boldsymbol{p})$

$=$

Frequentist approach

off-line

$$T(x, t, p) = \sum\sum \mathrm{T}_n(x)\, a_{nm}(t)\, \mathrm{P}_m(p)$$

$$X(x, t, p) = \sum\sum \mathrm{T}_n(x)\, a_{nm}(t)\, \frac{\mathrm{dP}_m}{\mathrm{d}p}(p)$$

on-line

- solution + cost function gradient
- 2 EDP $\rightsquigarrow$ 2 parametric solutions
- no acceleration of convergence

Bayesian approach

off-line

$$T(x, t, p) = \sum\sum \mathrm{T}_n(x)\, a_{nm}(t)\, \mathrm{P}_m(p)$$

on-line

- global solution
- 1 EDP $\rightsquigarrow$ 1 parametric solution
- no acceleration of acceptance rate

Outlooks

Outlooks

- Extension for multi-dimensional parameter... not easy!
- Extension for function estimation... how?
- Other ROM techniques available:

POD solution [7]:

$$T(x, t, p) = \sum_{n=1}^{N} \varphi_n(x)\, a_n(t, p),$$

where the time function basis $\left\{ a_n(t, p) \right\}_{n=1}^{N}$:

- is computed for a set of parameters (offline phase)
- is interpolated on a tangent space of GRASSMANN manifold for a new parameter [8] (online phase)

PGD solution[9–15] :

$$T(x, t, p) = \sum_{n=1}^{N} \varphi_n(x)\, a_n(t)\, f_n(p),$$

where the solution is computed at once by multiple projection of main Eq.

Outlooks

Other ROM techniques

MIM solution[1, 16, 17]

- *a posteriori* technique
- use experimental data to build the ROM
- solution is approximated by:

$$\frac{\partial a_n}{\partial t} = \boldsymbol{p}\, \mathcal{F}_{n\,n}\, a_n + \sum_{k=1}^{N_k} \mathcal{G}_{n\,k}\, \mathcal{Q}_k\,,$$

$$T(x_j, t) = \sum_{n=1}^{N} \varphi_{j\,n}\, a_n(t)\,,$$

where

- a_n are the internal states, **no** physical meaning,
- $\mathcal{Q}_k(t)$ are the N_k inputs of the system,
- the unknown parameter is extracted from the matrix identification.

Thank you for your attention

SAVE THE DATE!



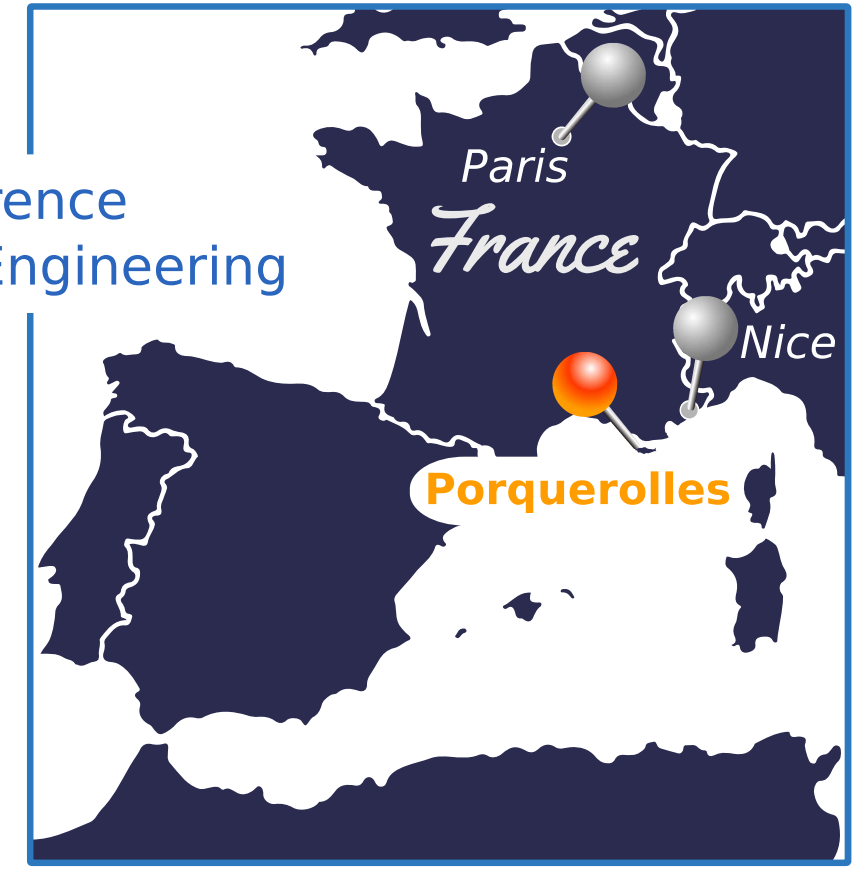
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12th International Conference
on Inverse Problems in Engineering

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Topics

1. Mathematical & statistical aspects:

- Theory and Methods
- Stability and error analysis
- Computational methods
- Model Identification, Reduction
- Machine learning for inversion
- Model Verification and Validation
- Uncertainty quantification

2. Design of Experiments:

- Optimal Design of Experiments
- Analysis of Actual Experimental Data
- Design under uncertainty
- Model and measurements
- Noise measurement & propagation
- Sensor modelling

3. Types of Inverse Problems:

- Physical properties estimation
- Boundary conditions estimation
- Initial conditions estimation
- Source term estimation
- Geometric characteristics estimation
- Application to multi-disciplinary engineering science:
  - Heat & mass Transfer
  - Solid & fluid Mechanics
  - Biology, Biochemistry & Medicine
  - Tomography & Inverse Scattering
  - Environmental Phenomena
  - Vibrations, Acoustics
  - Imaging
  - Nanoengineering
  - Food & Bioprocessing
  - Bio- & Medical-engineering
  - ... *non-exhaustive list*

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We hope to see you!



Some references I

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[3] J. Berger, H. Orlande, N. Mendes, and S. Guernouti. Bayesian inference for estimating thermal properties of a historic building wall. *Building and Environment*, 106:327–339, 2016.

[4] J. Berger, H. R. B. Orlande, and N. Mendes. Proper Generalized Decomposition model reduction in the Bayesian framework for solving inverse heat transfer problems. *Inverse Problems in Science and Engineering*, 25(2):260–278, 2017.

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