Simulations numériques d'instabilités thermo-convectives de fluides à seuil (modèle de Bingham) par Méthode Asymptotique Numérique

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Key issues to be overcome

- Visco-plastic (yield stress) modelling:
  - Augmented lagrangian techniques (Dean, Glowinski, Guidoboni, 2007)
  - Regularisation techniques (Bercovier & Engelman, 1980), (Frigaard & Nouar, 2005), etc.
- A reliable and computationally efficient coupled velocity-pressure formulation:
  - Discontinuous strain: only local approximations
  - Highly non-linear behaviour: challenging implementation of non-linear algorithm (tricky jacobian, etc.)
  - High computational costs in 3D (CPU ≈ $Neq.L_b^2$)
    - Cost_Augmented Lagrangian ≈ 16 x Cost_Regularization

Governing equations

- Conservation equations

$$\vec{\nabla} \cdot \vec{V} = 0$$

$$\vec{V} \cdot \nabla \vec{V} = -\vec{\nabla} p + \frac{1}{Re} \vec{\nabla} \cdot \bar{\bar{\tau}} + \frac{Ra}{Re^2\, Pr}\, \theta\, \vec{e_z}$$

$$\vec{V} \cdot \vec{\nabla} \theta = \frac{1}{RePr} \vec{\nabla} \cdot (\vec{\nabla} \theta)$$

- Constitutive equations: Bingham model

$$\begin{cases} \tau(\vec{V}) \leq Bn \Longleftrightarrow \dot{\gamma}(\vec{V}) = 0 \\ \tau(\vec{V}) > Bn \Longleftrightarrow \bar{\bar{\tau}}(\vec{V}) = \left[1 + \dfrac{Bn}{\dot{\gamma}(\vec{V})} \right] \bar{\bar{\dot{\gamma}}}(\vec{V}) \end{cases}$$

$$\tau(\vec{V}) = \left[\frac{1}{2} (\bar{\bar{\tau}} : \bar{\bar{\tau}}) \right]^{\frac{1}{2}} \quad ; \quad \dot{\gamma}(\vec{V}) = \left[\frac{1}{2} (\bar{\bar{\dot{\gamma}}} : \bar{\bar{\dot{\gamma}}}) \right]^{\frac{1}{2}}$$

$$Re = \frac{\rho V_{ref}, L_{ref}}{\mu_0} \quad ; \quad Ra = \frac{g \beta \Delta T_{ref} L_{ref}^3}{\nu \alpha} \quad ; \quad Pr = \frac{\nu}{\alpha} \quad ; \quad Bn = \frac{\tau_0 L_{ref}}{\mu_0 V_{ref}} \quad ; \quad \theta = \frac{T - T_{ref}}{\Delta T_{ref}}$$

Regularization techniques

$$\bar{\bar{\tau}}(\vec{V}) = \left[1 + \frac{Bn}{\dot{\gamma}_\eta(\vec{V})}\right]\bar{\bar{\dot{\gamma}}}(\vec{V})$$

- Simple regularization:

$$\dot{\gamma}_\eta(\vec{V}) = \left[\frac{1}{2}(\bar{\bar{\dot{\gamma}}} : \bar{\bar{\dot{\gamma}}})\right]^{\frac{1}{2}} + \eta$$

- Bercovier-Engelmann regularization:

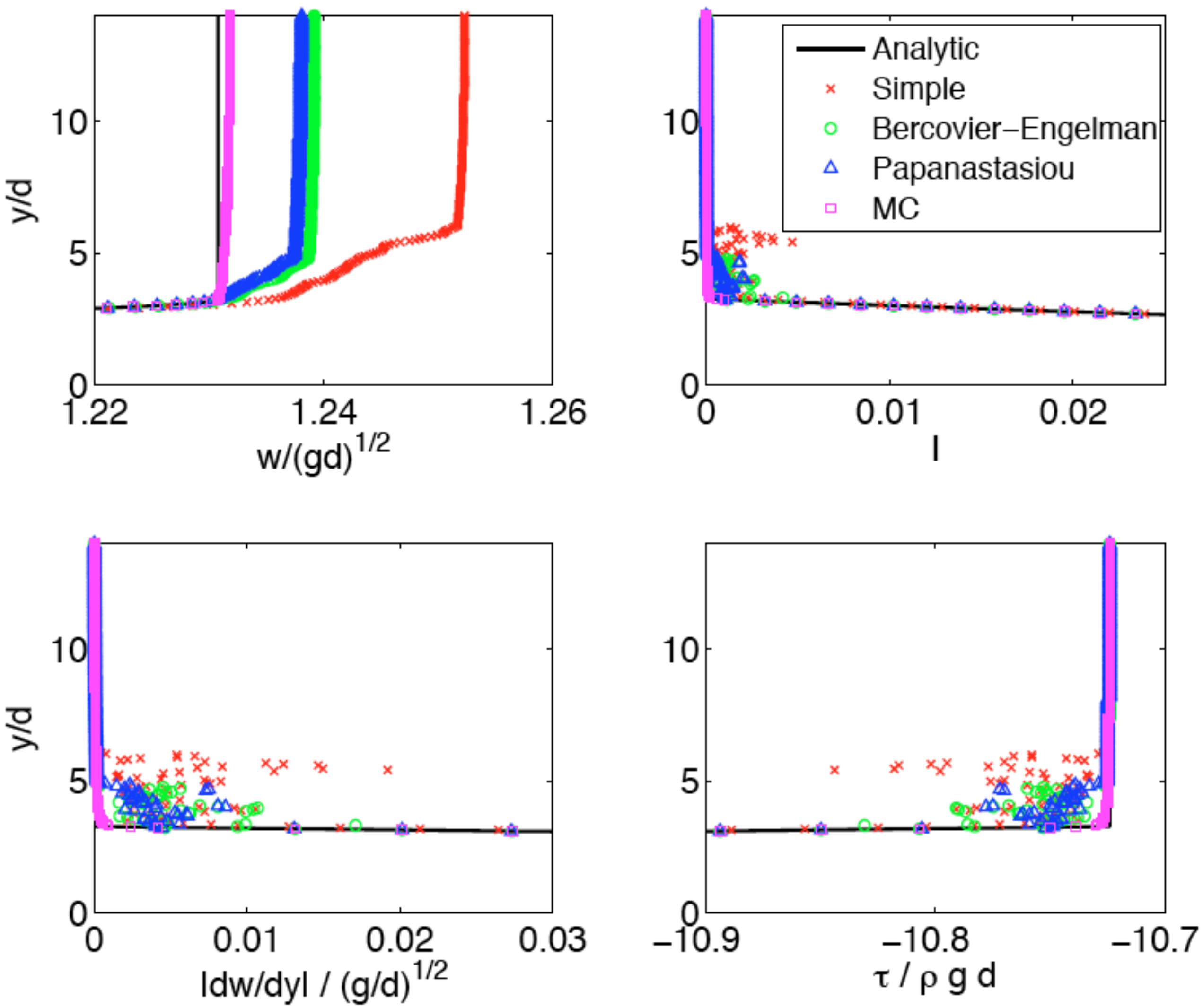
$$\dot{\gamma}_\eta(\vec{V}) = \left[\frac{1}{2}(\bar{\bar{\dot{\gamma}}} : \bar{\bar{\dot{\gamma}}}) + \eta^2\right]^{\frac{1}{2}}$$

- Papanastasiou regularization:

$$\frac{Bn}{\dot{\gamma}} = \frac{Bn(1 - e^{-\frac{\dot{\gamma}}{\eta}})}{\dot{\gamma}}$$

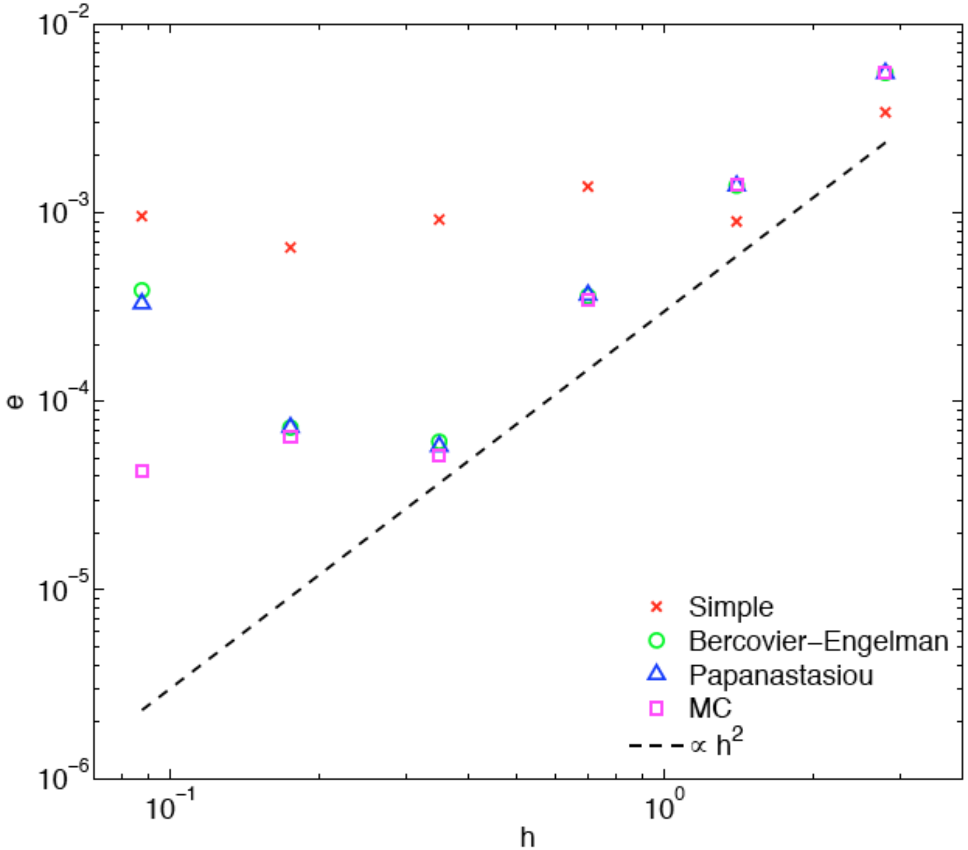
Regularization results (1/2)

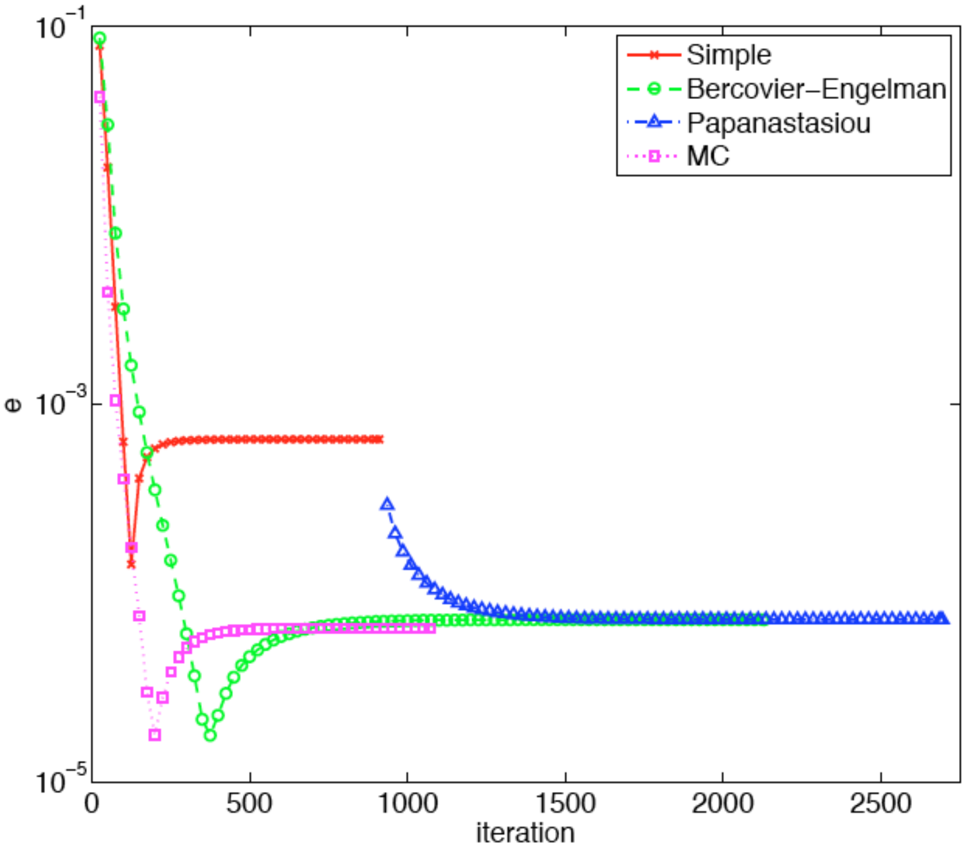
Vertical chute flow: Error localization (4/5)



→ All regularizations except the proposed one present spurious oscillations on $\frac{dw}{dy}$ and τ.

Regularization accuracy & perf





→ Simple regularization is the worst one in terms of error

→ The other regularizations have qualitatively the same behavior

→ Our method exhibits a better behavior for the finest mesh (e / 10)

→ Simple and proposed regularizations converge faster than BE and Papanastasiou

ANM continuation algorithm

- Bifurcation diagram usually consists in performing a sequence of two steps:
  - Compute base state associated to a given value of control parameters;
  - Compute linear stability of the given base state
- Continuation is performed with our steady state ANM continuation algorithm[1]:
  - Fully coupled velocity-pressure (Q2-Q-1 approx., H27 FE)
  - No stabilization

[1] B. Cochelin and M. Medale. Power series analysis as a major breakthrough to improve the efficiency of Asymptotic Numerical Method in the vicinity of bifurcations. J. Comput. Phys. Vol. 236, 594-607, 2013.

Asymptotic Numerical method (1)

General non-linear quadratic form:

$$R(U,\lambda) = L(U) + Q(U,U) - \lambda F = 0 \qquad (1)$$

Expand unknowns (U, λ) with respect to path parameter s:

$$\begin{cases} U(s) = U_0 + s\,U_1 + s^2\,U_2 + s^3\,U_3 + s^4\,U_4 + \ldots + s^n\,U_n & (2) \\ \lambda(s) = \lambda_0 + s\,\lambda_1 + s^2\,\lambda_2 + s^3\,\lambda_3 + s^4\,\lambda_4 + \ldots + s^n\,\lambda_n & (3) \\ (U_0,\lambda_0) \text{ been known.} \end{cases}$$

Parameterizations:

- Bifurcation type: $s = (U - U_0).U_1 + (\lambda - \lambda_0)\,\lambda_1$ (Cochelin, 1994)
- Minimal residual: $\lambda_n = \mathrm{Min}(\mathrm{Res}(n+1))$ (Lopez, 2000) (4)

Asymptotic Numerical Method (2)

Inserting (2-3) in (1, 4) and equating like powers of s:

Order 1
$$\begin{cases} L_t(U_1) = \lambda_1 F \\ < U_1, U_1 > + \lambda_1^2 = 1 \end{cases}$$

Order 2
$$\begin{cases} L_t(U_2) = \lambda_2 F - Q(U_1, U_1) \\ < U_1, U_2 > + \lambda_1 \lambda_2 = 0 \end{cases}$$

⋮

Order n
$$\begin{cases} L_t(U_n) = \lambda_n F - \sum_{r=1}^{n-1} Q(U_r, U_{n-r}) \\ < U_1, U_n > + \lambda_1 \lambda_n = 0 \end{cases}$$

FEM Spatial Discretization:

Order 1
$$\begin{cases} \left[K_t(U_0)\right]\{U_1\} = \lambda_1 \{F\} \\ \{U_1\}^t \{U_1\} + \lambda_1^2 = 1 \end{cases}$$

Order 2
$$\begin{cases} \left[K_t(U_0)\right]\{U_2\} = \lambda_2 \{F\} - \{FQ(1)\} \\ \{U_1\}^t \{U_2\} + \lambda_1 \lambda_2 = 0 \end{cases}$$

⋮

Order n
$$\begin{cases} \left[K_t(U_0)\right]\{U_n\} = \lambda_n \{F\} - \{FQ(n\text{-}1)\} \\ \{U_1\}^t \{U_n\} + \lambda_1 \lambda_n = 0 \end{cases}$$

Asymptotic Numerical Method (3)

Asymptotic expansion coefficients:

$$I_\tau = \int_\Omega \delta\bar{\bar{\varepsilon}} : \left[(1 + D_0)\dot{\bar{\bar{\gamma}}}_p + D_p \dot{\bar{\bar{\gamma}}}_0 + \sum_{r=1}^{p-1} D_r \dot{\bar{\bar{\gamma}}}_{(p-r)} \right] d\Omega$$

$$D_0 = \frac{Bn}{\dot{\gamma}_0} \; ; \quad D_1 = -\frac{D_0 \dot{\gamma}_1}{\dot{\gamma}_0} \; ; \quad D_p = -\frac{1}{\dot{\gamma}_0} \left[D_0 \dot{\gamma}_p + \sum_{r=1}^{p-1} D_r \dot{\gamma}_{(p-r)} \right]$$

$$\dot{\gamma}_0 = \left[\frac{1}{2} (\dot{\bar{\bar{\gamma}}}_0 : \dot{\bar{\bar{\gamma}}}_0) + \eta^2 \right]^{\frac{1}{2}} \; ; \; \dot{\gamma}_1 = \frac{\dot{\bar{\bar{\gamma}}}_0 : \dot{\bar{\bar{\gamma}}}_1}{2\dot{\gamma}_0} \; ; \; \dot{\gamma}_p = \frac{1}{2\dot{\gamma}_0} \left[\dot{\bar{\bar{\gamma}}}_0 : \dot{\bar{\bar{\gamma}}}_p + \frac{1}{2} \sum_{r=1}^{p-1} \dot{\bar{\bar{\gamma}}}_r : \dot{\bar{\bar{\gamma}}}_{(p-r)} - \sum_{r=1}^{p-1} \dot{\gamma}_r \dot{\gamma}_{(p-r)} \right]$$

$$I_\tau = \int_\Omega \delta\bar{\bar{\varepsilon}} : \left[(1 + D_0)\dot{\bar{\bar{\gamma}}}_p - \frac{\dot{\bar{\bar{\gamma}}}_0}{\dot{\gamma}_0} \left\{ \frac{D_0}{2\dot{\gamma}_0} \left(\dot{\bar{\bar{\gamma}}}_0 : \dot{\bar{\bar{\gamma}}}_p + \frac{1}{2} \sum_{r=1}^{p-1} \dot{\bar{\bar{\gamma}}}_r : \dot{\bar{\bar{\gamma}}}_{(p-r)} - \sum_{r=1}^{p-1} \dot{\gamma}_r \dot{\gamma}_{(p-r)} \right) + \sum_{r=1}^{p-1} D_r \dot{\gamma}_{(p-r)} \right\} + \sum_{r=1}^{p-1} D_r \dot{\bar{\bar{\gamma}}}_{(p-r)} \right] d\Omega$$

Bifurcation point detection

- In the ANM geometrical power series arises close to BP

- Geometric progression detection algorithm:

$$
\begin{aligned}
&\text{for } n-3 \le p \le n-1 \\
&\qquad \alpha_p = (U_p \cdot U_n)/(U_n \cdot U_n) \\
&\qquad U_p^{\perp} = U_p - \alpha_p U_n \\
&\text{if } \sum_{p=n-3}^{n-2} \left((\alpha_p^{1/(n-p)} - \alpha_{n-1})/\alpha_{n-1} \right)^2 < \varepsilon_{gp_1} \\
&\text{and } \textstyle\sum_{p=n-3}^{n-1} \|U_p^{\perp}\|/\|U_p\| < \varepsilon_{gp_2}
\end{aligned}
$$

- If test satisfied (colinearity, proportionality), one computes
  - its common ratio: $\alpha = \alpha_{n-1} \approx 1/r$
  - its scale factor: $U_{t_1}^{\perp} = U_n/\alpha^n$

Bifurcation point computation branch switching

- Solution at bifurcation point:

$$U_{BP} = U(s = \tfrac{1}{\alpha}) = U_0 + \tfrac{1}{\alpha}\hat{U}_1 + \tfrac{1}{\alpha^2}\hat{U}_2 + \cdots + \tfrac{1}{\alpha^{n-1}}\hat{U}_{n-1}$$

with $\hat{U}_p = U_p - \alpha^p U_{t_1}^{\perp}$

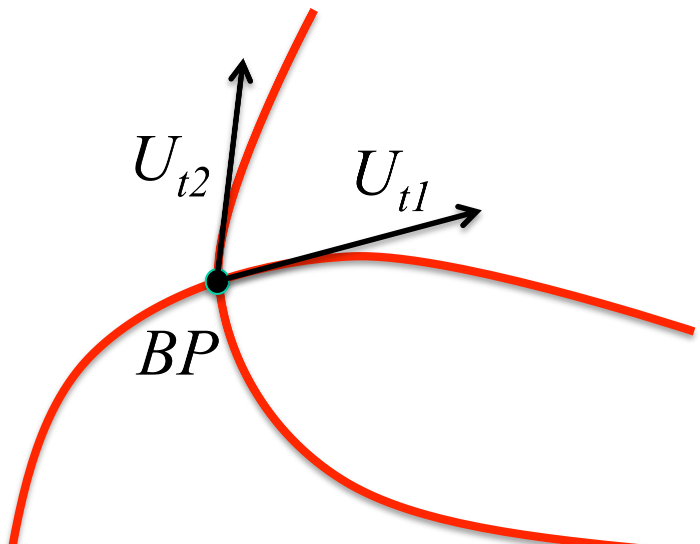
- Tangent vector to branch 1:

$$U_{t_1} = \frac{\partial U}{\partial s}(s = \frac{1}{\alpha}) = \hat{U}_1 + \frac{2}{\alpha}\hat{U}_2 + \cdots + \frac{n-1}{\alpha^{n-2}}\hat{U}_{n-1}$$

- Tangent vector to branch 2:

$$U_{t_2} = \beta\, U_{t_1} + \gamma\, U_{t_1}^{\perp}$$

$$\psi^T R^c_{,UU} U_{t_2} U_{t_2} = 0$$



Confined Rayleigh-Benard

- Parallelepipedic computational domain: L/h=10; l/h=4

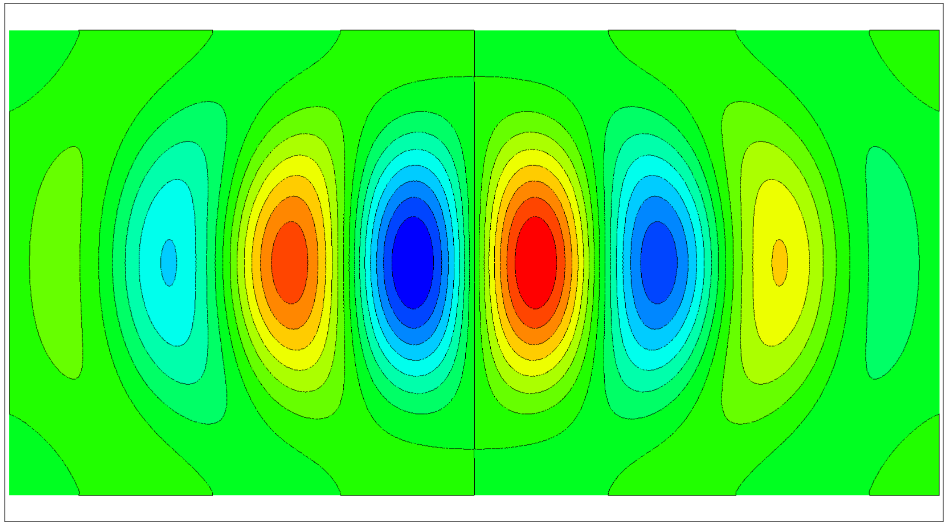
- Spatial discretization: 180x72x18 H27 Q2 FE (7 747 060 dof)

- ANM parameters: $10 \le n \le 50$; $\delta = 10^{-9}$; $\varepsilon_{gp1} = 10^{-3}$ and $\varepsilon_{gp1} = 10^{-6}$;
  1 NR end-of-step correction if Res > 10^{-6}

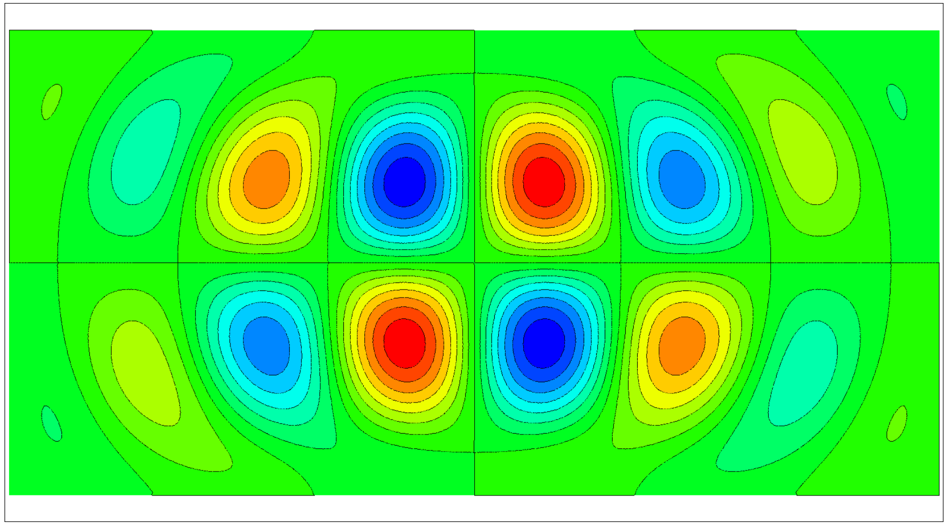
- HPC: Petsc + MUMPS + BULL Bullx S6010 supercomputer
(128 cores, 2 To RAM)
  - fact. time: 45 mn (on 64 cores)
  - av. cont. step ($n \approx 50$): 50 mn (on 64 cores)

Rayleigh-Benard in a box

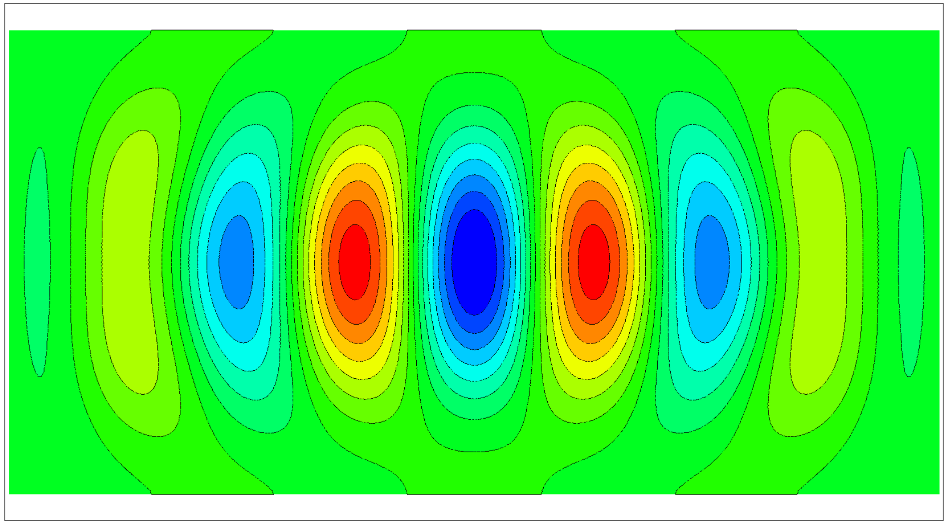
Newtonian fluid: Pr=9



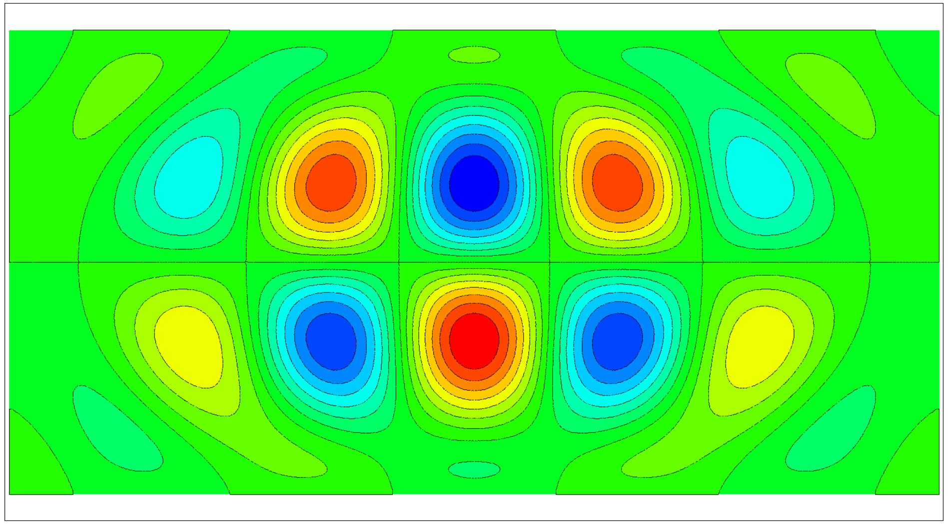
1st bifurcation mode: Ra_{c1}=1747.4



3rd bifurcation mode: Ra_{c3}=1773.9



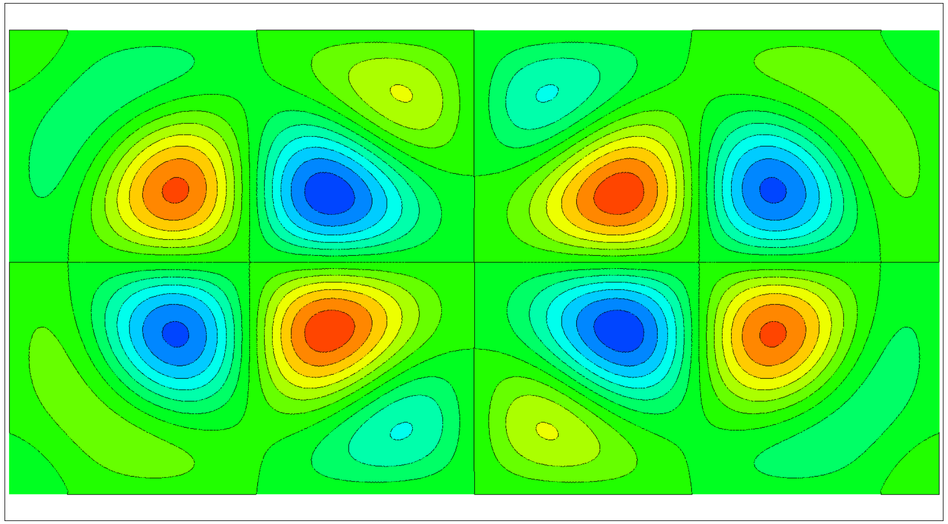
2nd bifurcation mode: Ra_{c2}=1748.8



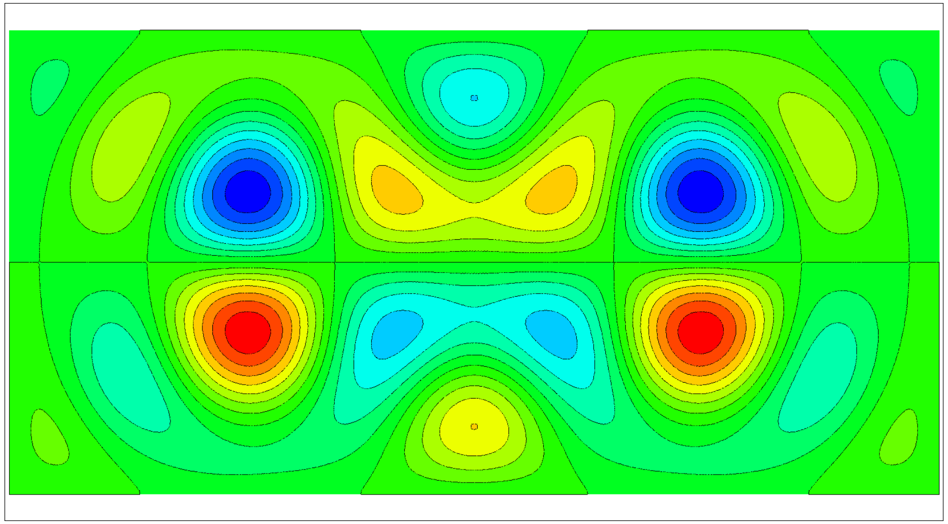
4th bifurcation mode: Ra_{c4}=1805.7

Rayleigh-Benard in a box

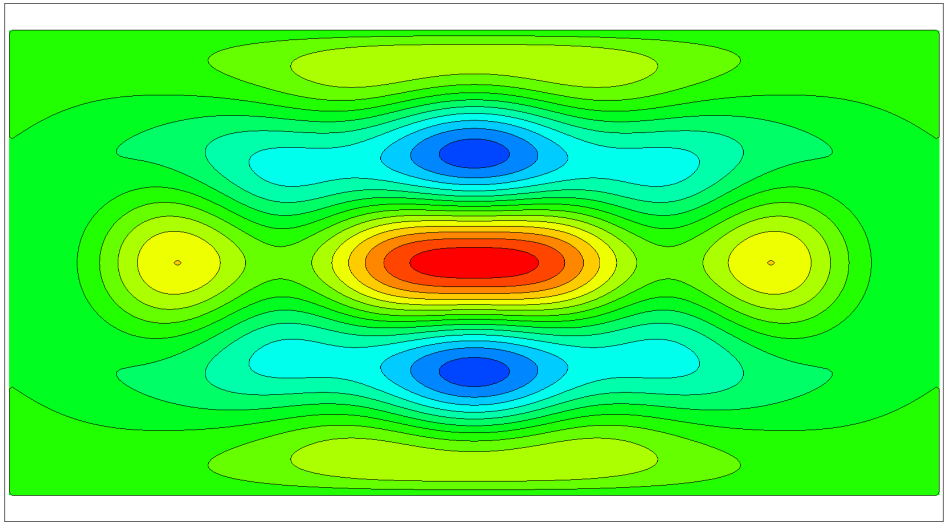
Newtonian fluid: Pr=9



5th bifurcation mode: Ra_{c5}=1810.9



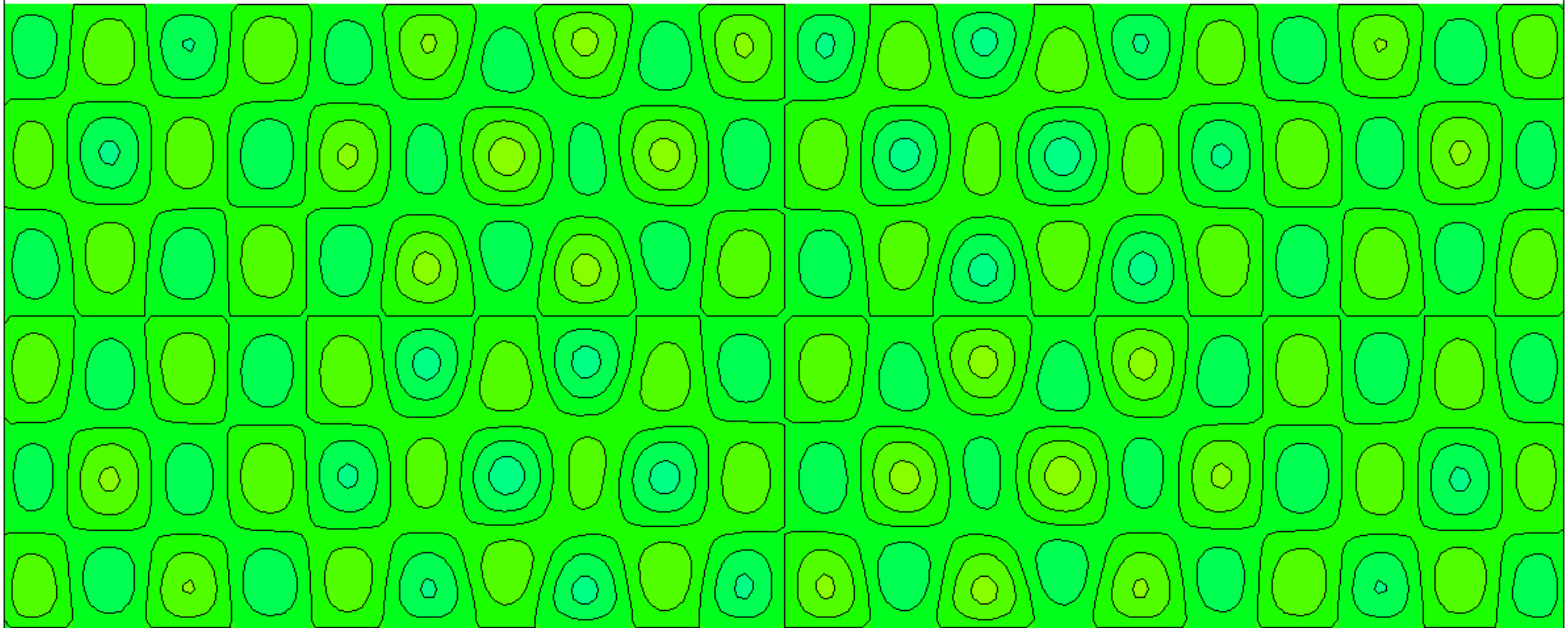
6th bifurcation mode: Ra_{c6}=1812.9



7th bifurcation mode: Ra_{c7}=1817.2

Rayleigh-Benard in a box

Bingham fluid: Pr=9, Bn=1



Summary

- 3D Steady-State Solver for incompressible fluid flows
- Continuation method (ANM predictor - NR corrector) to perform bifurcation diagram and accurately locate bifurcation points
- Efficient and scalable parallel implementation for problems up to several millions of dof
- Rayleigh-Benard in a box (Γ_x=10, Γ_y=4)
  - Detailed results for Newtonian fluids (Pr=9)
  - Preliminary results for a Bingham fluid (Pr=9, Bn=1)
- Future works:
  - Nature of bifurcations, linear stability analysis
  - High multiplicity bifurcation points, Hopf bifurcation

Asymptotic Numerical Method (6)

ANM : Perturbation method + F.E.M.

- Optimal step length (Cochelin, 1994, 2003) :

$$s_{opt} = (\varepsilon \ \|F_1\| \,/\, \|F_{nl}(n+1)\| \,)^{(1/n+1)}$$

- Enables us to describe one part of the branch
- Loop with new starting point ($U(s=s_{opt})$, $\lambda(s=s_{opt})$)

High order predictor method, Newton based corrector

Main features:

- Parameter free continuation technique (ε, n)
- Analytical representation of quadratic non-linearities
- Power series contain substantial informations